

# Representations of Symmetric Groups and Free Probability

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We consider representations of symmetric groups  $S_q$  for large  $q$ . We give the asymptotic behaviour of the characters when the corresponding Young diagrams, rescaled by a factor  $q^{-1/2}$ , converge to some prescribed shape. This behaviour can be expressed in terms of the free cumulants for a probability measure associated with the limit shape of the diagram. We also show that the basic operations of representation theory, like taking tensor products, restriction, or induction, have a limiting behavior which can be described using the free probability theory of D. Voiculescu. © 1998 Academic Press

## INTRODUCTION

Almost everything seems to be known about representation theory of symmetric groups, indeed there exists algorithms, such as the Murnaghan–Nakayama formula or the Littlewood–Richardson rule, which allow one to solve the most basic problems, like computing the value of characters, decomposing induced representations, or restrictions of representations. While being efficient theoretical tools, these formulas become quite cumbersome when the symmetric groups, and the corresponding Young diagrams, become large. In this case, representations contain a large amount of information, and we only expect to understand their statistical structure. This topic, asymptotic theory of the representations of symmetric groups, has been studied, especially by A. Vershik and S. Kerov, in connection with representation theory of  $S_\infty$ , the inductive limit of finite symmetric groups, see e.g. [VK2] or the survey papers [Ve1], [Ve2]. They have shown that the theory of factor representations of  $S_\infty$ , which is well understood, can be recovered as the limit, when  $q \rightarrow \infty$ , of that of the symmetric groups on  $q$  objects  $S_q$ , when the Young diagrams are suitably rescaled. More precisely, the indecomposable normalized characters of  $S_\infty$  are in one to one correspondence with pairs of sequences  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k \geq \dots \geq 0$  and  $\beta_1 \geq \beta_2 \geq \dots \geq \beta_k \geq \dots \geq 0$  (the “asymptotic frequencies”) satisfying  $\Sigma \alpha_j + \Sigma \beta_j \leq 1$ , and the corresponding normalized character

$\chi_{\alpha, \beta}$  is the pointwise limit, as  $q \rightarrow \infty$ , of a sequence  $\chi_{\lambda_q}$ , of normalized characters of  $S_q$ , associated with Young diagrams  $\lambda_q$ , with  $q$  boxes. The sequence  $\lambda_q$ ;  $q \geq 1$  satisfies  $c_k(\lambda_q)/q \rightarrow \alpha_k$ ,  $l_k(\lambda_q)/q \rightarrow \beta_k$  as  $q \rightarrow \infty$ , where  $c_k$  (resp.  $l_k$ ) denotes the length of the  $k$ th column (resp. line) of the diagram. In particular, the sequences of diagrams with zero asymptotic frequencies converge to the character of the regular representation of  $S_\infty$ .

In this paper we shall consider the asymptotic properties of representations of  $S_q$  corresponding to “balanced” Young diagrams, by which we mean Young diagrams with  $q$  boxes, whose largest column and largest line are of the order  $O(\sqrt{q})$ . In this case the asymptotic frequencies are zero, and the character, evaluated on a fixed, non trivial conjugacy class, converges to zero as  $q \rightarrow \infty$ , so that we are looking at some infinitesimal neighborhood of the regular representation of  $S_\infty$ . We shall see that the speed of convergence to zero depends in a precise way on the asymptotic shape of the diagrams. Indeed let  $\lambda_q$ ;  $q \geq 1$  be a sequence of Young diagrams with  $q$  boxes, such that the diagrams, rescaled by a factor  $q^{-1/2}$ , converge to some limit shape  $\omega$ , as  $q \rightarrow \infty$  (see Section 1 for precise definitions and statements), then the normalized character  $\chi_{\lambda_q}$ , evaluated on a fixed conjugacy class has the asymptotic behavior

$$\chi_{\lambda_q}(\sigma) = C_\sigma(\omega) q^{-|\sigma|/2} + O(q^{-|\sigma|/2-1})$$

where  $|\sigma|$  is the smallest number  $k$ , needed to write  $\sigma$  as a product of  $k$  transpositions, and the numbers  $C_\sigma(\omega)$  depend only on the limit shape and on the conjugacy class of  $\sigma$ , i.e. the numbers  $k_2, k_3, \dots$  of cycles of  $\sigma$  of orders 2, 3, .... The coefficients  $C_\sigma(\omega)$  are multiplicative with respect to  $\sigma$ , i.e. they satisfy  $C_{\sigma_1 \sigma_2}(\omega) = C_{\sigma_1}(\omega) C_{\sigma_2}(\omega)$  when  $\sigma_1$  and  $\sigma_2$  are permutations which disjoint supports, and their values on cycles are the free cumulants of a probability measure naturally associated with the limit shape  $\omega$ . The limit shapes and the associate probability measures have already appeared in the work by S. Kerov [Ke1], [Ke2], where the probability measure is called the transition measure of the diagram. In this paper we shall show that the transition measure has a natural spectral interpretation in terms of the Jucys–Murphy elements in the algebra of the symmetric group. The notion of free cumulants of a probability measure comes from the theory of free probability of D. Voiculescu, especially in the combinatorial treatment of this theory which has been given by R. Speicher, see e.g. [VDN] for an introduction to free probability theory, and [Sp1], [Sp2] for the combinatorial side. The Jucys–Murphy elements have already been used to construct asymptotic models of semi-circular systems in [Bil]. In our asymptotic analysis of characters of the symmetric group, the connection between the free cumulants and the symmetric group is obtained through a bijection between the lattice of non-crossing partitions, which plays a

prominent role in [Sp1], and a certain subset of the symmetric group. This bijection has already been studied in [Kr], [Bi3], [Bi4].

The asymptotic behavior of characters already indicates that free probability theory should be relevant for understanding representations of large symmetric groups, with balanced Young diagrams. As we shall see, this is indeed correct, and we show below that the basic operations of representation theory have a limiting behavior for large balanced Young diagrams, which can be described entirely in terms of free probability theory. We shall treat the three basic operations, namely taking tensor products, restrictions, or outer products of representations. For each of these operations, involving large balanced Young diagrams, we shall show that in the decomposition of the resulting representations, most Young diagrams appearing are close to a specific shape, which can be explicitly computed. Rather than state our results in full generality here (see in Sections 1.4 to 1.6, and 3.1 below), we shall explain informally how they look like on a specific example. Consider a large integer  $L$  and the representation  $[\lambda]$  of  $S_{L^2}$  with square Young diagram  $\lambda = (L, L, L, \dots, L)$ , as in Fig. 1. Application of Theorem 1.6.1 below then shows that when  $L$  is large enough, the irreducible components appearing in the decomposition of the outer product  $[\lambda] \circ [\lambda]$  (which is a representation of  $S_{2L^2}$ ), have Young diagrams which are almost all close to the triangular shape of Fig. 2.

Analogously, in the decomposition of the restriction of  $[\lambda]$  to  $S_{L^2/2}$  (or  $S_{(L^2-1)/2}$  if  $L$  is odd), almost all diagrams become close to the triangular shape of Fig. 3.

We have also results for more general asymptotic shapes, and for the problems of decomposing Kronecker tensor products of representations. All these results can be expressed in terms of free probability theory, once

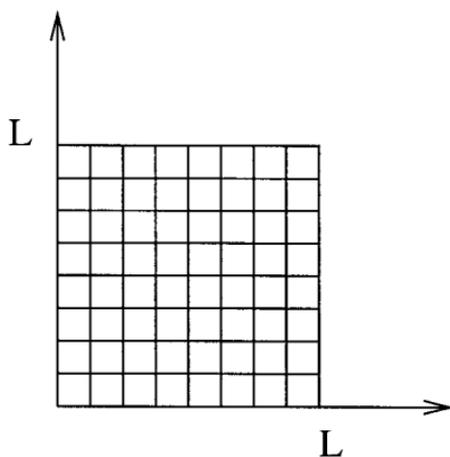


FIGURE 1

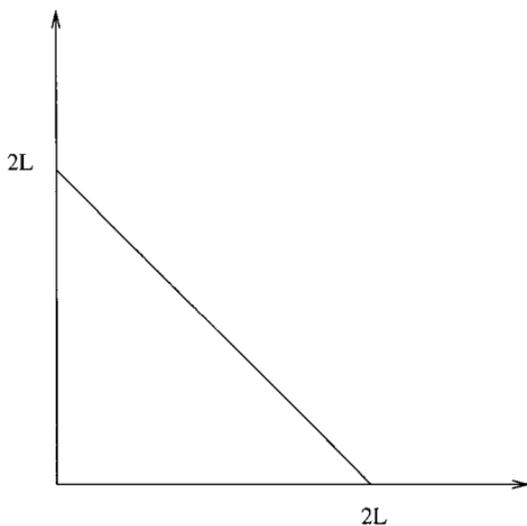


FIGURE 2

we have defined the probability measure associated to a diagram. The problem of restriction to a subgroup involves the compression of a non-commutative random variable with a free projection, while the decomposition of outer products involves the free convolution of two measures. For example, the results involving the diagrams above correspond to the fact that the free convolution square of the symmetric Bernoulli distribution on  $\{-L, +L\}$  is the arcsine distribution with density  $1/(\pi \sqrt{4L^2 - x^2})$  on  $[-2L, 2L]$ , while the compression of the symmetric Bernoulli distribution on  $\{-L, +L\}$ , by a free projection of trace  $1/2$  is the arcsine distribution on  $[-L, L]$ . For the decomposition of tensor products, almost all diagrams appearing are close to the shape corresponding to the semi-circular distribution. This shape is the limit shape of Young diagrams taken at random with the Plancherel measure, as shown in [LS], [VK1].

Connections between representation theory and free probability have already been noticed in [Bi2], where representations of large unitary groups were considered. There, the free probability appeared through

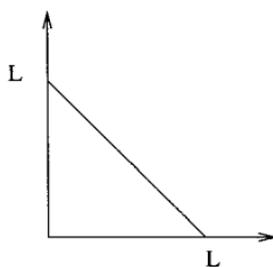


FIGURE 3

large matrices, which give rise to free random variables. In the present paper, we use large matrix theory only at one point, when we consider the problem of decomposing outer products, i.e. the Littlewood–Richardson rule. This problem is similar to the one we considered in [Bi2], but we want to stress that, although the results we obtain here have a certain similarity, we are in fact dealing with different normalizations, and the two series of results cannot be related directly. In the present paper we use a more combinatorial approach to these problems.

This paper is organized as follows. In the first part, we introduce the basic notations on symmetric groups and Young diagrams, in order to state our main results. The second section is devoted to free probability theory. We recall the main facts, with emphasis on the combinatorial aspects. In the third section we introduce the correspondence between limit shapes of Young diagrams and probability measures, which we use to restate the main results in an appropriate form. Section 4 is devoted to proving the result on the asymptotic behavior of characters, while the problems of decomposing tensor products and restrictions of representations are dealt with in Sections 5 and 6. Finally, in the last four sections we consider the outer products of representations. The proof there is much more involved than in the other cases, and we have to make a detour through representation theory of general linear groups, using Schur–Weyl’s duality.

## 1. NOTATIONS AND STATEMENT OF THE RESULTS

### 1.1. *Symmetric Groups and Young Diagrams*

Let  $q$  be a positive integer, in the following we shall denote by  $S_q$  the symmetric group on  $q$  objects; most of the time, we will assume that the  $q$  objects are the integers  $1, 2, \dots, q$  and we shall denote by  $(ij)$  the transposition exchanging  $i$  and  $j$ , and more generally by  $(i_1 i_2 \dots i_n)$  cyclic permutations of order  $n$ .

If  $\sigma \in S_q$ , let  $c(\sigma)$  be the number of cycles of this permutation. It is well known that  $|\sigma| := q - c(\sigma)$  is the smallest number  $n$  such that  $\sigma$  can be written as a product of  $n$  transpositions, thus  $d(\sigma, \sigma') = |\sigma' \sigma^{-1}|$  defines a left- and right-invariant distance on  $S_q$ . We shall also call support of a permutation the set of points which are not fixed by the permutation, and denote by  $s(\sigma)$  the number of elements in the support of  $\sigma$ .

Finally we shall denote by  $2^{k_2} 3^{k_3} \dots$  the conjugacy class of permutations with  $k_2$  cycles of order two,  $k_3$  cycles of order 3, etc... Note that one has  $|\sigma| = k_2 + 2k_3 + \dots$  and  $s(\sigma) = 2k_2 + 3k_3 + \dots$  when the conjugacy class of  $\sigma$  is  $2^{k_2} 3^{k_3} \dots$ .

A Young diagram is a nonincreasing sequence of nonnegative integers with a finite sum. The usual representation of a Young diagram is as a union of finitely many boxes. See e.g. the diagram 3, 2, 2, 1, 0, ... (Fig. 4).

We denote by  $\mathcal{Y}_q$  the set of Young diagrams with  $q$  boxes, and  $\mathcal{Y} = \bigcup_{q=1}^{\infty} \mathcal{Y}_q$ .

**DEFINITION.** Let  $A > 1$ , we say that a Young diagram  $\lambda \in \mathcal{Y}_q$  is  $A$ -balanced if  $\text{width}(\lambda) := \max(\text{col}(\lambda), \text{line}(\lambda)) \leq A \sqrt{q}$ , where  $\text{col}(\lambda)$  (resp.  $\text{line}(\lambda)$ ) is the number of columns (resp. lines) of  $\lambda$ .

Note that for all  $\lambda \in \mathcal{Y}_q$ , one always has  $\text{col}(\lambda) \text{line}(\lambda) \geq q$ , so that for all  $A$ -balanced diagrams  $\lambda \in \mathcal{Y}_q$  one has  $\sqrt{q/A} \leq \text{col}(\lambda)$ ,  $\text{line}(\lambda) \leq A \sqrt{q}$ .

If  $\lambda$  is a Young diagram with  $q$  boxes, we shall denote by  $[\lambda]$  the associated (equivalence class of) irreducible representation of  $S_q$ , and by  $\chi_\lambda$  its normalized character. As usual, normalized means that the value of the character at the identity permutation is 1.

In Sections 1 and 3 we shall discuss continuous Young diagrams as well as probability measures associated to them. Our exposition will be based on that of S. Kerov [Ke1], [Ke2], to which we refer for more details.

Making the change of coordinates  $(x, y) \mapsto (x - y, x + y)$  in the graphical representation of the Young diagram, we obtain the equivalent picture of a piecewise linear function  $\omega$ , with slopes  $\pm 1$ , and local minima and maxima occurring at two interlacing sequences of integer points

$$x_1 < y_1 < x_2 < \dots < y_{n-1} < x_n$$

For the diagram of Fig. 4, these sequences are  $(-3, -1, 2, 4)$  and  $(-2, 1, 3)$  respectively (see Fig. 5).

For a Young diagram  $\lambda$ , identified with a function in this way, we shall denote by  $|\lambda|$  the number of boxes of  $\lambda$ , which is equal to  $\frac{1}{2} \int_{\mathbb{R}} \lambda(x) - |x| dx$ . Note that  $\text{width}(\lambda)$  is the smallest  $c \geq 0$  such that  $\lambda(x) = |x|$  for all  $x$  with  $|x| \geq c$ .

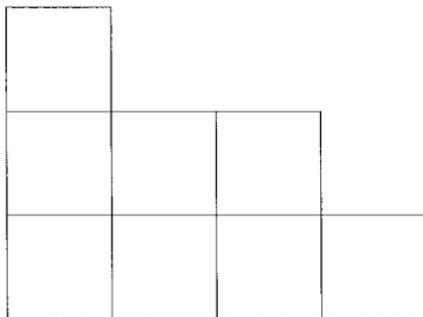


FIGURE 4

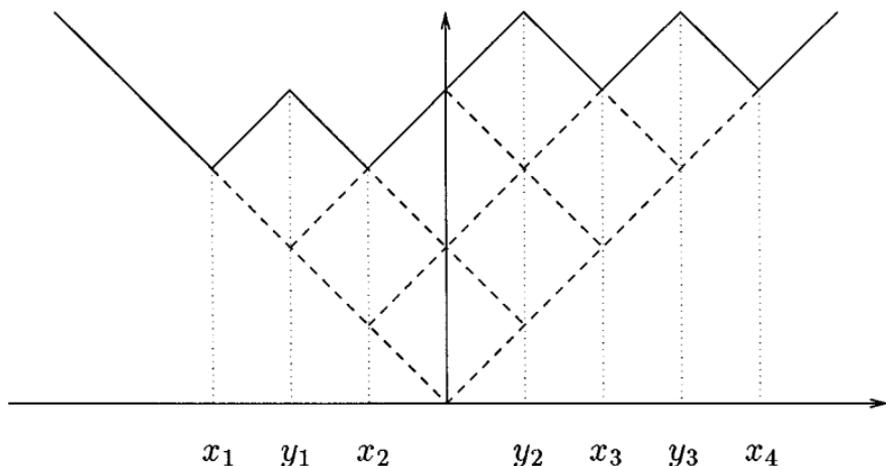


FIGURE 5

We can thus embed the set  $\mathcal{Y}$  of Young diagrams in the space  $\mathcal{CY}$  (continuous Young diagrams) of functions  $\omega: \mathbb{R} \rightarrow \mathbb{R}$  satisfying

- (1)  $|\omega(u_1) - \omega(u_2)| \leq |u_1 - u_2|$  for all  $u_1, u_2 \in \mathbb{R}$
- (2)  $\omega(u) = |u|$  for sufficiently large  $|u|$ .

### 1.2. Generating Function of a Diagram

For a continuous diagram  $\omega \in \mathcal{CY}$ , we consider the functions  $\sigma(u) = (\omega(u) - |u|)/2$ ;  $u \in \mathbb{R}$ , and

$$G_\omega(z) = \frac{1}{z} \exp \int_{\mathbb{R}} \frac{1}{x-z} \sigma'(x) dx$$

where  $z$  is a complex variable. Note that, since  $\omega$  is Lipschitz, the function  $\sigma'$  is defined almost everywhere on  $\mathbb{R}$ , is zero outside a compact interval, and its absolute value is bounded by 1, thus the function  $G_\omega$  is defined and holomorphic outside a compact interval of  $\mathbb{R}$ . The function  $G_\omega$  is called the generating function of the diagram  $\omega$  in Kerov's papers. For a Young diagram  $\lambda$ , with associated interlacing sequences  $(x_i)_{1 \leq i \leq n}$  and  $(y_i)_{1 \leq i \leq n-1}$ , one has

$$G_\lambda(z) = \frac{\prod_{i=1}^{n-1} (z - y_i)}{\prod_{i=1}^n (z - x_i)}$$

For a general  $\omega \in \mathcal{CY}$ , the function  $G_\omega$  has an expansion into a series convergent in a neighborhood of infinity

$$G_\omega(z) = \frac{1}{z} + \sum_{n=2}^{\infty} a_n(\omega) z^{-n-1}$$

where  $a_2 = q$  in the case of a Young diagram with  $q$  boxes. We can thus define the inverse, for composition, of  $G_\omega$  in a neighborhood of  $\infty$ , which is given by a power series

$$K_\omega(z) = \frac{1}{z} + \sum_{n=1}^{\infty} C_n(\omega) z^{n-1}$$

convergent in a neighborhood of zero. We shall call the coefficients  $C_n$  the free cumulants of the diagram (we shall explain this terminology and its connection to free probability in Section 3 below). One has  $C_1(\omega) = 0$  for all  $\omega \in \mathcal{CY}$ , and  $C_2(\lambda) = |\lambda|$  for all Young diagrams.

Note that the effect of a scaling transformation, changing the function  $\omega$  into  $u \mapsto T^{-1}\omega(Tu)$  is to multiply the coefficients  $a_n$  and  $C_n$  by  $T^{-n}$  (it is clear that  $\mathcal{CY}$  is invariant under such scaling transformations).

The following is a simple consequence of basic results of free probability theory, and will be proved below in Section 3.1.

**THEOREM 1.2.** *Let  $\omega, \omega' \in \mathcal{CY}$ , and  $0 < t < 1$ , then*

(1) *There exists a unique diagram  $\pi_t(\omega) \in \mathcal{CY}$  such that  $C_n(\pi_t(\omega)) = t^{n-1}C_n(\omega)$  for all  $n \geq 2$ .*

(2) *There exists a unique diagram  $\omega \boxplus \omega' \in \mathcal{CY}$  such that  $C_n(\omega \boxplus \omega') = C_n(\omega) + C_n(\omega')$  for all  $n \geq 2$ .*

**EXAMPLES.** Consider the square diagram  $\omega(x) = \max(|x|, 2L - |x|)$  (obtained from the one in Fig. 1. of the introduction). Then one has  $G_\omega(z) = z/(z-L)(z+L)$  and  $C_{2p}(\omega) = (-1)^{p+1} L^{2p}((2p)!/(p+1)! p!)$ ,  $C_{2p+1} = 0$ . A short computation shows that  $\omega \boxplus \omega(x) = \max(|x|, 2L)$  and  $\pi_{1/2}(\omega)(x) = \max(|x|, L)$ , which correspond to the diagrams in Fig. 2. and Fig. 3. in the introduction.

Another example is provided by the limit curve of Young diagrams with respect to Plancherel measure (see [LS] and [VK1]),

$$\begin{aligned} \Omega(x) &= \frac{2}{\pi} \left( x \arcsin \frac{x}{2} + \sqrt{4 - x^2} \right) && \text{for } |x| \leq 2, \\ &= |x| && \text{for } |x| \geq 2. \end{aligned}$$

See the Fig. 6.

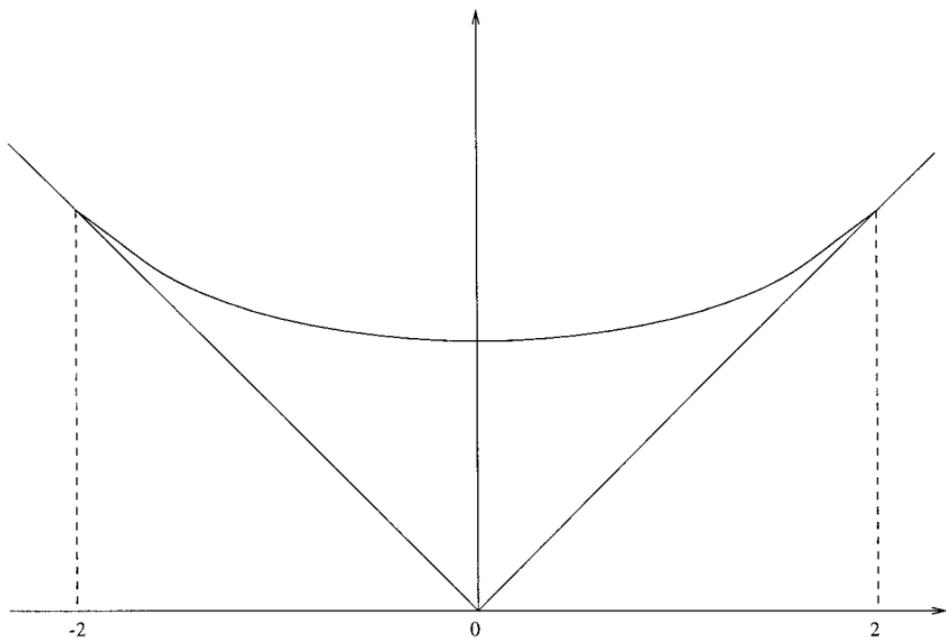


FIG. 6 The curve  $\Omega$ .

The rescaled functions  $\Omega_s(x) = s^{1/2}\Omega(s^{-1/2}x)$  satisfy  $C_2(\Omega_s) = s$  and  $C_n(\Omega_s) = 0$  for  $n \geq 3$ . It follows that for all  $s, s' \geq 0$ , and  $0 < t < 1$ , one has  $\Omega_s \boxplus \Omega_{s'} = \Omega_{s+s'}$  and  $\pi_t(\Omega_s) = \Omega_{ts}$ .

### 1.3. Asymptotics of Characters

The first application of free cumulants that we give is to the asymptotic behavior of characters of  $S_q$  when  $q \rightarrow \infty$ , for balanced diagrams.

**THEOREM 1.3.** *For all  $A > 1$  and  $m$  positive integer, there exists a constant  $K > 0$  such that, for all  $A$ -balanced Young diagrams  $\lambda$ , and all permutations  $\sigma \in S_{|\lambda|}$  satisfying  $|\sigma| \leq m$ , one has*

$$\left| \chi_\lambda(\sigma) - \prod_{c|\sigma} |\lambda|^{-|c|-1} C_{|c|+2}(\lambda) \right| \leq K |\lambda|^{-1-|\sigma|/2}$$

where the product is over the disjoint cycles  $c$  of the permutation  $\sigma$ .

Note that, since  $C_2(\lambda) = |\lambda|$ , only the non-trivial cycles contribute to the product (recall that  $|c| = k - 1$  if  $c$  is a cycle of order  $k$ ). In order to see that Theorem 1.3 gives the right order of magnitude for the characters, consider a sequence  $\lambda_N$  of  $A$ -balanced diagrams satisfying  $|\lambda_N| \rightarrow \infty$ , and such that the rescaled diagrams  $|\lambda_N|^{-1/2} \lambda_N(|\lambda_N|^{1/2} u)$  converge uniformly to some continuous diagram  $\omega$ , then one has  $C_n(\lambda_N) \sim |\lambda_N|^{n/2} C_n(\omega)$ , and

thus  $\chi_{\lambda_N}(\sigma) = |\lambda_N|^{-|\sigma|/2} \prod_{c|\sigma} C_{|c|+2}(\omega) + O(|\lambda_N|^{-|\sigma|/2-1})$ , as  $N \rightarrow \infty$ , by Theorem 1.3.

Observe that Theorem 1.3 implies the following asymptotic factorization property for characters.

**COROLLARY 1.3.** *For all  $A > 1$ , and  $m$  positive integer, there exists a constant  $K > 0$  such that, for all  $A$ -balanced Young diagrams  $\lambda$ , and all permutations  $\sigma, \tau \in S_{|\lambda|}$  with disjoint supports, satisfying  $|\sigma|, |\tau| \leq m$ , one has*

$$|\chi_\lambda(\sigma\tau) - \chi_\lambda(\sigma)\chi_\lambda(\tau)| \leq K |\lambda|^{-|\sigma|/2 - |\tau|/2 - 1}$$

This is to be compared with the factorization property for characters of  $S_\infty$  or related groups (see [VK2] and [Vo3]).

*Remark.* The constant appearing in Theorem 1.3 (and in the Corollary), can be made explicit. In fact by a slightly more careful analysis than the one we carry out below, the theorem can be extended in several ways. On one hand, one can deal with Young diagrams satisfying a condition of the kind  $\text{width}(\lambda) \leq A(q) \sqrt{q}$  with  $A(q) = o(q^\varepsilon)$  for every  $\varepsilon > 0$ , on the other hand it is possible to have estimates uniform over all permutations satisfying  $|\sigma| = O((\log q)^{1-\varepsilon})$  for some  $\varepsilon > 0$ . The only difficulty in obtaining these extensions is to keep track of all the constants involved. Since the proof of the basic result we present is already quite cumbersome we will not try to go into these matters.

### 1.4. Asymptotic of Tensor Products

We now turn to the problem of decomposing the Kronecker tensor product of two balanced representations (we denote by  $[\lambda] \otimes [\mu]$  the Kronecker tensor product of representations).

**THEOREM 1.4.1.** *For every  $A > 1$ , and  $\varepsilon, \delta > 0$ , there exists  $q_0$  such that for all  $q \geq q_0$ , and all  $A$ -balanced Young diagrams  $\lambda, \mu \in \mathcal{Y}_q$ , the subspace of the representation  $[\lambda] \otimes [\mu]$ , which is spanned by all the isotypic components of type  $[v]$ , satisfying*

$$\sup_{x \in \mathbb{R}} |v(x) - \Omega_q(x)| \leq \varepsilon \sqrt{q},$$

*has dimension larger than  $(1 - \delta) \dim([\lambda] \otimes [\mu])$ .*

The function  $\Omega_q$  is defined in Example 1.2. Besides Theorem 1.4.1 we also have the following concentration result for characters.

**THEOREM 1.4.2** (zero rule for character of tensor products). *For every  $A > 1$ , all  $\varepsilon, \delta > 0$ , and integer  $n > 0$ , there exists  $q_0$  such that for all  $q \geq q_0$ ,*

and all  $A$ -balanced Young diagrams  $\lambda, \mu \in \mathcal{Y}_q$ , the subspace of the representation  $[\lambda] \otimes [\mu]$ , which is spanned by all the isotypic components of type  $[v]$ , satisfying

$$|\chi_v(\sigma)| \leq \varepsilon q^{-|\sigma|/2}, \quad \text{for all } |\sigma| \leq n,$$

has dimension larger than  $(1 - \delta) \dim([\lambda] \otimes [\mu])$ .

### 1.5. Asymptotics of Restrictions

If  $p < q$ , we denote by  $[\lambda] \downarrow S_p \times S_{q-p}$  the restriction of the representation  $[\lambda]$  of  $S_q$  to the subgroup  $S_p \times S_{q-p}$ . For  $v_1 \in \mathcal{Y}_p$ , and  $v_2 \in \mathcal{Y}_{q-p}$  we denote by  $[v_1] \# [v_2]$  the tensor product of  $[v_1]$  and  $[v_2]$ , which is an irreducible representation of  $S_p \times S_{q-p}$ .

**THEOREM 1.5.1.** *For every  $A > 1$ , and  $\alpha \in ]0, 1/2]$ , all  $\varepsilon, \delta > 0$ , there exists  $q_0$  such that for all  $q \geq p \geq q_0$ , with  $\alpha q \leq p \leq (1 - \alpha)q$ , and all  $A$ -balanced Young diagrams  $\lambda \in \mathcal{Y}_q$ , the subspace of the representation  $[\lambda] \downarrow S_p \times S_{q-p}$ , which is spanned by all the isotypic components  $[v_1] \# [v_2]$  satisfying*

$$\sup_{x \in \mathbb{R}} |v_1(x) - \pi_{(p+1)/(q+1)}(\lambda)(x)| \leq \varepsilon \sqrt{q} \quad \text{and}$$

$$\sup_{x \in \mathbb{R}} |v_2(x) - \pi_{(q-p+1)/(q+1)}(\lambda)(x)| \leq \varepsilon \sqrt{q}$$

has dimension larger than  $(1 - \delta) \dim([\lambda] \downarrow S_p \times S_{q-p}) = (1 - \delta) \dim([\lambda])$ .

As in Section 1.4, we can give a version of this result involving characters of cycles.

**THEOREM 1.5.2.** *For every  $A > 1$ , and  $\alpha \in ]0, 1/2]$ , all  $\varepsilon, \delta > 0$ , and integer  $n > 0$ , there exists  $q_0$  such that for all  $q \geq p \geq q_0$ , with  $\alpha q \leq p \leq (1 - \alpha)q$ , and all  $A$ -balanced Young diagrams  $\lambda, \mu \in \mathcal{Y}_q$ , the subspace of the representation  $[\lambda] \downarrow S_p \times S_{q-p}$ , which is spanned by all the isotypic components  $[v_1] \# [v_2]$  satisfying*

$$|\chi_{v_1}((12 \cdots m)) - \chi_\lambda((12 \cdots m))| \leq \varepsilon q^{-(m-1)/2}, \quad \text{and}$$

$$|\chi_{v_2}(12 \cdots m) - \chi_\lambda((12 \cdots m))| \leq \varepsilon q^{-(m-1)/2} \quad \text{for } m \leq n,$$

has dimension larger than  $(1 - \delta) \dim([\lambda] \downarrow S_p \times S_{q-p}) = (1 - \delta) \dim([\lambda])$ .

1.6. *Asymptotics of Induction*

Finally we treat the case of the induction of representations. Recall that, given representations  $[\lambda]$  and  $[\mu]$  of  $S_q$  and  $S_r$  respectively,  $[\lambda] \circ [\mu]$  denotes the outer product of the representations  $[\lambda]$  and  $[\mu]$ , namely the representation of  $S_{q+r}$  induced by the representation  $[\lambda] \# [\mu]$  of  $S_q \times S_r \subset S_{q+r}$ .

**THEOREM 1.6.1.** *For every  $A > 1$ , and  $\varepsilon, \delta > 0$ , there exists  $q_0$  such that for all  $q \geq q_0$ , and all Young diagrams  $\lambda$  and  $\mu$ , satisfying  $\text{width}(\lambda), \text{width}(\mu) \leq A \sqrt{q}$ , and  $|\lambda|, |\mu| \geq q_0$ , the subspace of the representation  $[\lambda] \circ [\mu]$ , which is spanned by all the isotypic components  $[v]$ , satisfying*

$$\sup_{x \in \mathbb{R}} |v(x) - \lambda \boxplus \mu(x)| \leq \varepsilon \sqrt{q},$$

*has dimension larger than  $(1 - \delta) \dim([\lambda] \circ [\mu])$ .*

**THEOREM 1.6.2** (sum rule for induction). *For every  $A > 1$ , all  $\varepsilon, \delta > 0$ , and  $n$  positive integer there exists  $q_0$  such that for all  $q \geq q_0$ , and all Young diagrams  $\lambda, \mu$  satisfying  $\text{width}(\lambda), \text{width}(\mu) \leq A \sqrt{q}$  and  $|\lambda|, |\mu| \geq q_0$ , the subspace of the representation  $[\lambda] \circ [\mu]$ , which is spanned by all the isotypic components  $[v]$  satisfying*

$$\begin{aligned} &| |v|^m \chi_v((12 \cdots m)) - |\lambda|^m \chi_\lambda((12 \cdots m)) \\ &\quad - |\mu|^m \chi_\mu((12 \cdots m)) | \leq \varepsilon q^{(m+1)/2} \end{aligned}$$

*for all  $2 \leq m \leq n$ , has dimension larger than  $(1 - \delta) \dim([\lambda] \circ [\mu])$ .*

1.7. *Some Remarks*

One can give explicit versions of the constants involved in these results, as will be clear from the proofs, however, since the bounds that one obtains in this way are probably far from being optimal, we did not bother to state and prove the results with these explicit formulas, which would be more cumbersome.

We have chosen to state our results in terms of uniform estimates on the class of  $A$ -balanced Young diagrams. We could alternatively have considered sequences of  $A$ -balanced Young diagrams  $\lambda_N$  with  $|\lambda_N| \rightarrow \infty$ , such that the rescaled diagrams  $|\lambda_N|^{-1/2} \lambda_N(|\lambda_N|^{1/2} u)$  converge uniformly to some function  $\omega \in \mathcal{CY}$ , as in Section 1.3. We leave the reader to state the results in this setting which is perhaps easier to grasp. More important, is the fact that the condition that the diagram be  $A$ -balanced can be weakened. Indeed, it will be clear from the proofs we give that we could replace this condition by  $\text{width}(\lambda) \leq \phi(|\lambda|)$  and  $|C_n(\lambda)| \leq c_n |\lambda|^{n/2}$  for some function  $\phi$  satisfying  $\phi(q) = o(q^{1/2+\eta})$  for every  $\eta > 0$ , and some sequence

$c_n > 0$ . This would allow us to consider sequences of diagrams such that  $|\lambda_N|^{-1/2} \lambda_N(|\lambda_N|^{1/2} u)$  converge uniformly to some function  $\omega$ , satisfying  $\omega(u) \geq |u|$  for all  $u \in \mathbb{R}$ , and  $\omega(u) - |u| \rightarrow 0$  as  $|u| \rightarrow \infty$ . The only difference in the results would be that estimates of the form  $O(q^{-n-1/2})$  would be replaced by estimates of the form  $o(q^{-n-1/2+\eta})$  for every  $\eta > 0$ . The proof of such results is essentially the same as the one we give, but with heavier notations so we keep the more straightforward version given above.

Finally, extensions of Theorems 1.4.1, 1.5.1, and 1.6.1 along the lines we presented in the remark at the end of Section 1.3, can be obtained by the same methods. For example in Theorem 1.6.1, there exists a function  $\phi(q) = o(\sqrt{q})$  such that the condition  $\sup_{x \in \mathbb{R}} |v(x) - \lambda \boxplus \mu(x)| \leq \varepsilon \sqrt{q}$  can be replaced by the condition  $\sup_{x \in \mathbb{R}} |v(x) - \lambda \boxplus \mu(x)| \leq \phi(q)$ . This boils down to finding a good modulus of continuity for the map  $\omega \mapsto m_\omega$  to be defined in Section 3 below. Again, proving these stronger forms would make the proofs longer, so we stick to the version given above.

### 1.8. Connection with the Hook Walk

The results of the papers [Ke1] and [Ke2], where the semi-circular distribution, and the complex Burger's equation appear in the limiting behavior of the hook walk, have suggested to the author of the present paper that there should exist a connection between representations of symmetric groups and free probability. Let us explain in a few words how Kerov's results can be interpreted in terms of ours. Kerov considers the mean value of the Young diagrams obtained from an initial Young diagram  $\lambda \in \mathcal{Y}_p$  by inducing  $[\lambda]$  from  $S_p$  to  $S_q$  (this is the representation theoretic content of the "hook walk"), he then shows that after rescaling, so that  $q$  becomes a continuous parameter  $t$ , the generating function of the mean diagram, called  $R$  in Kerov's paper, satisfies the complex Burger's equation  $(\partial R / \partial t) + R(\partial R / \partial z) = 0$ . Inducing  $[\lambda]$  from  $S_p$  to  $S_q$  gives the same decomposition as inducing  $[\lambda] \# \text{reg}$ , where  $\text{reg}$  is the regular representation of  $S_{q-p}$ , and moreover, for large  $q-p$ , almost all the diagrams occurring in  $\text{reg}$  are close to the limit curve  $\Omega_{q-p}$  so that, by our result Theorem 1.6.1, after rescaling, almost all Young diagrams occurring in  $[\lambda] \circ \text{reg}$  will have their transition measure close to the free convolution of the initial measure with a semi-circular distribution. It is well known that the semi-group of free convolution by a semi-circular distribution is governed by the complex Burger's equation (see e.g. [VDN]), so that we recover Kerov's result.

## 2. FREE PROBABILITY THEORY

We recall some facts and definitions from the theory of free random variables. Our main references are [VDN], and [BV].

### 2.1. Non-commutative Probability Spaces

Let  $\mathcal{A}$  be a von Neumann algebra and  $\varphi$  a normal faithful trace on  $\mathcal{A}$ . Following [VDN], [BV], we call non-commutative probability space such a couple  $(\mathcal{A}, \varphi)$ . We denote by  $\mathcal{A}_{sa}$  the subspace of self-adjoint elements of  $\mathcal{A}$ . If  $X \in \mathcal{A}_{sa}$ , the distribution of  $X$  in the state  $\varphi$  is the unique probability measure  $m_X$ , with compact support on  $\mathbb{R}$ , such that  $\varphi(f(X)) = \int_{\mathbb{R}} f(x) m_X(dx)$  for any bounded Borel function  $f$  on  $\mathbb{R}$ .

### 2.2. Free Random Variables

Let  $I$  be a set of indices, and  $\mathcal{B}_i$ , for  $i \in I$ , be unital von Neumann sub-algebras of  $\mathcal{A}$ . They are called free if one has  $\varphi(\alpha_1 \cdots \alpha_n) = 0$  each time  $\varphi(\alpha_j) = 0$  for all  $j = 1, \dots, n$  and  $\alpha_j \in \mathcal{B}_{i_j}$  for some indices  $i_1 \neq i_2 \neq \cdots \neq i_n$ .

Let  $l$  and  $m$  be probability measures with compact support on  $\mathbb{R}$ , then there exists a non-commutative probability space  $(\mathcal{A}, \varphi)$  and self-adjoint elements  $X, Y$  in  $\mathcal{A}$  with respective distributions  $l$  and  $m$ , such that  $X$  and  $Y$  are free, i.e. the von Neumann algebras generated by their spectral projections are free. The distribution of  $X + Y$  depends only on  $l$  and  $m$ , it is called the free additive convolution of  $l$  and  $m$  and is denoted by  $l \boxplus m$ . This defines a symmetric and associative binary operation on the set of probability measures with compact support on  $\mathbb{R}$ .

Similarly, if  $m$  is a probability measure and  $0 < t < 1$ , let  $X$  be a self-adjoint element in some non-commutative probability space  $(\mathcal{A}, \varphi)$ , with distribution  $m$ , and  $\Pi \in \mathcal{A}$  be a self-adjoint projection of trace  $t$ , which is free with  $X$ . The free compression of  $X$  by  $\Pi$  is the self-adjoint element  $\Pi X \Pi$ . Its distribution in the non-commutative probability space  $(\Pi \mathcal{A} \Pi, (1/t) \varphi_{|\Pi \mathcal{A} \Pi})$  which we denote by  $\pi_t(m)$ , depends only on  $m$  and  $t$ .

### 2.3 Computation of Free Convolution

The explicit computation of the free additive convolution of two measures on  $\mathbb{R}$  can be effected in an analytic way, as follows. Recall that the Cauchy transform of a probability measure  $m$  on  $\mathbb{R}$  is

$$G_m(\zeta) = \int_{\mathbb{R}} \frac{1}{\zeta - x} m(dx)$$

This defines an analytic function on  $\mathbb{C} \setminus \mathbb{R}$ , such that  $G_m(\bar{\zeta}) = \overline{G_m(\zeta)}$ , furthermore, this function uniquely determines the measure  $m$ . If  $m$  has compact support, then  $G$  is analytic and invertible in a neighborhood of  $\infty$ . Call  $K_m$  this right inverse, and let  $R_m(z) = K_m(z) - (1/z)$ , then for all probability measure  $l, m$ , with compact support, one has

$$R_{l \boxplus m}(z) = R_l(z) + R_m(z)$$

in a neighborhood of 0. Since  $1 \boxplus m$  is determined by the restriction of  $R_{1 \boxplus m}$  to any neighborhood of 0, this characterizes completely the measure  $1 \boxplus m$ .

Note that the operation  $\boxplus$  is continuous on the space of probability measures with compact support, endowed with the topology of weak convergence.

These results were established in [Vo1]. For extension to arbitrary probability measures see [BV].

The computation of  $\pi_t(m)$  is analogous, indeed, one has

$$R_{\pi_t(m)}(z) = R_m(tz)$$

which determines completely the measure  $\pi_t(m)$ , see e.g. [NS].

#### 2.4. Combinatorial Approach to Freeness

Another approach to freeness, which is combinatorial in nature and relies on properties of non-crossing partitions, has been proposed by R. Speicher. We shall briefly recall some of the main facts now, coming from [Sp1] and [Sp2]. For some further references concerning the use of non-crossing partitions in free probability, see [N], [NS].

Let  $S$  be a totally ordered set. A partition of the set  $S$  is said to have a crossing if there exists a quadruple  $(i, j, k, l) \in S^4$ , with  $i < j < k < l$ , such that  $i$  and  $k$  belong to some class of the partition and  $j$  and  $l$  belong to another class. If a partition has no crossing, it is called non-crossing. The set of all non-crossing partitions of  $S$  is denoted by  $NC(S)$ , it is a lattice with respect to the dual refinement order (such that  $\pi \leq \sigma$  if  $\pi$  is a finer partition than  $\sigma$ ).

When  $S = \{1, \dots, n\}$ , with its natural order, we will use the notation  $NC(n)$ .

Let  $(\mathcal{A}, \varphi)$  be a non-commutative probability space. For each integer  $n \geq 1$ , let  $\psi^{(n)}$  be an  $n$ -linear form on  $\mathcal{A}$ . We shall forget the superscript  $n$  when it is clear which  $n$  is considered. For any  $n \geq 1$ , and  $\pi \in NC(n)$ , define an  $n$ -linear map  $\psi[\pi]$  on  $\mathcal{A}$  by

$$\psi[\pi](a_1, \dots, a_n) = \prod_{V \in \pi} \psi(a_V)$$

where  $a_V = (a_{j_1}, \dots, a_{j_k})$  if  $V = \{j_1, \dots, j_k\}$  is a class of the partition  $\pi$ , with  $j_1 < j_2 < \dots < j_k$ . In particular  $\psi[1_n] = \psi^{(n)}$  if  $1_n$  is the partition with one class. If  $\psi(a_1, \dots, a_n) = \varphi(a_1 \cdots a_n)$  we shall denote by  $\varphi[\pi]$  the corresponding  $n$ -linear maps. Obviously one has  $|\varphi[\pi](a_1, \dots, a_n)| \leq \prod_{j=1}^n \|a_j\|$ .

The non-crossing cumulants of  $\varphi$  are the family  $R^{(n)}$  of  $n$ -linear maps defined by

$$R^{(n)}(a_1, \dots, a_n) = \sum_{\pi \in NC(n)} Moeb(\pi) \varphi[\pi](a_1, \dots, a_n)$$

where  $Moeb$  is the Möbius function on  $NC(n)$ . The Möbius function is multiplicative on the set  $\bigcup_{n \geq 1} NC(n)$  of all non-crossing partitions, which means that for all  $\pi$  one has  $Moeb(\pi) = \prod_{V \in \pi} Moeb(1_{|V|})$ , furthermore  $Moeb(1_n) = (-1)^{n-1} c_{n-1}$  where  $c_{n-1} = (2n-2)!/n!(n-1)!$  are the Catalan numbers. Since  $c_{n-1} \leq 4^n$ , one has  $|Moeb(\pi)| \leq 4^n$  for  $\pi \in NC(n)$ , so that  $|R(a_1, \dots, a_n)| \leq 4^{2n} \prod_{j=1}^n \|a_j\|$ , since there are less than  $4^n$  elements in  $NC(n)$ . We denote by  $R[\pi]$  the  $n$ -linear forms corresponding to  $R$ . The formula giving  $R$  in terms of  $\varphi$  can be inverted, namely one has, for all  $a_1, \dots, a_n \in \mathcal{A}$ ,

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} R[\pi](a_1, \dots, a_n) \tag{2.4.1}$$

More generally, for any  $\pi \in NC(n)$ , one has

$$\varphi[\pi](a_1, \dots, a_n) = \sum_{\sigma \in NC(n), \sigma \leq \pi} R[\sigma](a_1, \dots, a_n) \tag{2.4.2}$$

### 2.5. Non-crossing Cumulants and Freeness

We take again the notations of 2.4. The connection between non-crossing cumulants and freeness is the following result of Section 4 of [Sp1].

**PROPOSITION 2.5.** *Let  $(\mathcal{B}_i)_{i \in I}$  be free subalgebras of  $\mathcal{A}$ , and  $a_1, \dots, a_n \in \mathcal{A}$  be such that  $a_j$  belongs to some  $\mathcal{B}_{i_j}$  for each  $j \in \{1, 2, \dots, n\}$ . Then  $R(a_1, \dots, a_n) = 0$  if there exists some  $j$  and  $k$  with  $i_j \neq i_k$ .*

This property has the following consequence for the computation of moments. Let  $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{A}$  be two free subalgebras, and  $a_1, \dots, a_n \in \mathcal{A}$  such that  $a_j$  belongs to  $\mathcal{B}_1$  or  $\mathcal{B}_2$  for each  $j$ . Let  $S_1 \cup S_2$  be a partition of  $\{1, \dots, n\}$  into two subsets such that  $a_j \in \mathcal{B}_1$  if  $j \in S_1$  and  $a_j \in \mathcal{B}_2$  if  $j \in S_2$ . A pair  $(\pi_1, \pi_2)$  of partitions of  $S_1$  and  $S_2$  determines a partition  $\pi_1 \cup \pi_2$  of  $S_1 \cup S_2$ . Let  $\psi_1$  and  $\psi_2$  be two families of  $n$ -linear functionals, and suppose that  $\pi_1 \cup \pi_2$  is non-crossing, we define an  $n$ -linear form

$$\psi_1 \cup \psi_2[\pi_1 \cup \pi_2](a_1, \dots, a_n) = \prod_{V \in \pi_1} \psi_1(a_V) \prod_{V \in \pi_2} \psi_2(a_V)$$

If  $\pi \in NC(S_1)$  let  $\pi^c$  be the maximal non-crossing partition of  $S_2$  such that  $\pi \cup \pi^c$  is non-crossing. One has then, by (2.4.1),

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} R[\pi](a_1, \dots, a_n)$$

By Proposition 2.5, the only elements of  $NC(n)$  which contribute to the sum are the partitions of the form  $\pi_1 \cup \pi_2$  where  $\pi_i \in NC(S_i)$  for  $i=1, 2$ , hence we have

$$\begin{aligned} \varphi(a_1 \cdots a_n) &= \sum_{\pi_1 \in NC(S_1)} \sum_{\substack{\pi_2 \in NC(S_2) \\ \pi_1 \cup \pi_2 \in NC(n)}} R \cup R[\pi_1 \cup \pi_2](a_1, \dots, a_n) \\ &= \sum_{\pi_1 \in NC(S_1)} \sum_{\substack{\pi_2 \in NC(S_2) \\ \pi_2 \leq \pi_1^c}} R \cup R[\pi_1 \cup \pi_2](a_1, \dots, a_n) \end{aligned}$$

Applying now (2.4.2), we see that

$$\varphi(a_1 \cdots a_n) = \sum_{\pi_1 \in NC(S_1)} R \cup \varphi[\pi_1 \cup \pi_1^c](a_1, \dots, a_n) \quad (2.5)$$

This formula comes from Section 3.4 of [Sp2]. We have reproduced the derivation of Speicher for the reader's convenience.

## 2.6. Combinatorics of Free Convolution

The connection between the combinatorics of Sections 2.4 and 2.5, and the analytic approach to free convolution of Section 2.3 is the following. Let  $X \in \mathcal{A}_{sa}$  have distribution  $m(dx)$ , then one has

$$\int_{\mathbb{R}} x^n m(dx) = \varphi(X^n) = \sum_{\pi \in NC(n)} R[\pi](X, \dots, X)$$

The map  $\pi \mapsto R[\pi](X, \dots, X)$  is a multiplicative function on  $\bigcup_{n \geq 1} NC(n)$ , determined by the values of  $R[1_n](X, \dots, X)$ , for  $n=1, 2, \dots$ , and one has

$$R_m(z) = \sum_{n=1}^{\infty} z^{n-1} R[1_n](X, \dots, X)$$

The coefficients of the expansion of  $R_m$  are called the free cumulants of the measure  $m$ , and we denote them by  $C_n(m)$ .

## 2.7. Non-crossing Partitions and Permutations

There is a useful connection between non-crossing partitions and permutations which we shall review in this section, see e.g. [Kr], [Bi4] for more details.

Let  $n$  denote a positive integer, and  $\tau$  and  $\sigma \in S_n$  be such that  $|\sigma| + |\sigma^{-1}\tau| = |\tau|$ , then the partition of  $\{1, \dots, n\}$  induced by the cycles of  $\sigma$  is finer than that of  $\tau$  (see e.g. Section 1 of [Bi4]).

Let now  $\tau$  be the cycle  $(12 \dots n)$  and  $\sigma$  be as above, then the partition of  $\{1, \dots, n\}$  induced by the cycles of  $\sigma$  is non-crossing and furthermore, for each  $j \in \{1, \dots, n\}$ , the element  $\sigma(j)$  is the one in the same component as  $j$  which comes first in the list  $\tau(j), \tau^2(j), \tau^3(j) \dots$ . Conversely, if  $\pi$  is a non-crossing partition of  $\{1, \dots, n\}$  let the permutation  $t(\pi)$  of  $\{1, \dots, n\}$  be defined by the preceding algorithm, i.e.  $t(\pi)(j)$  is the first element in the list  $\tau(j), \tau(j+1), \dots$  which is in the same class for  $\pi$  as  $j$ , then  $t(\pi)$  satisfies  $|t(\pi)| + |t(\pi)^{-1}\tau| = |\tau|$ , thus  $t$  is a bijection between  $NC(n)$  and the subset of all  $\sigma \in S_n$  satisfying the condition  $|\sigma| + |\sigma^{-1}\tau| = |\tau|$ . In this bijection, the components of  $\pi \in NC(n)$  correspond to the disjoint cycles of  $t(\pi)$ . The map  $\sigma \mapsto \sigma^{-1}\tau$  is a bijective map of  $t(NC(n))$ , which gives a bijective map on  $NC(n)$  called the Kreweras map, denoted by  $K$ . There is a geometric description of the Kreweras map. Put the  $n$  points  $1, \dots, n$  in cyclic order on a circle, and add a new point  $j'$  at the middle of  $j$  and  $j+1 \pmod{n}$  for each  $j$ , then represent a non-crossing partition  $\pi$  by joining each point to the next point for cyclic order, which lies in the same component, with a straight line segment. Then  $\pi$  is a non-crossing partition if and only if segments do not intersect inside the circle. The Kreweras complement of  $\pi$ ,

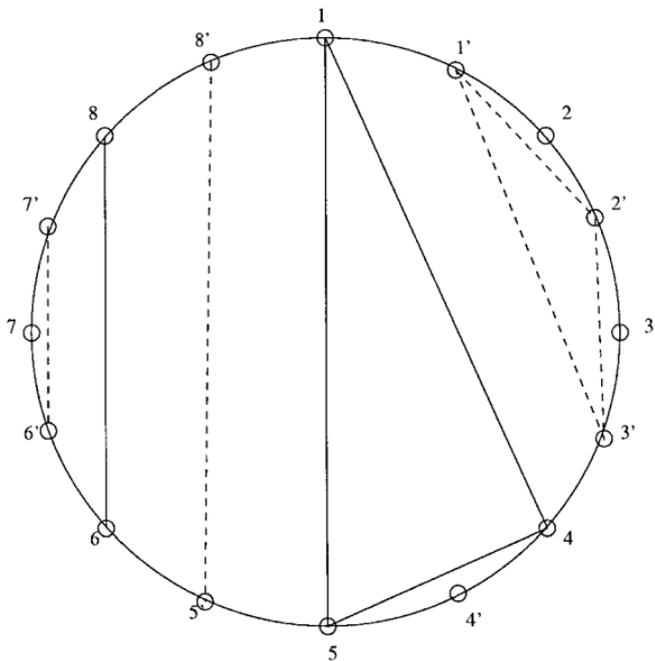


FIGURE 7

identified with a partition of  $\{1', 2', \dots, n'\}$ , is then the maximal non-crossing partition which does not cross  $\pi$ , i.e. such that  $\pi \cup \pi'$  is a non-crossing partition of  $\{1, 1', 2, 2', \dots, n, n'\}$ . See Fig. 7 for the case  $\tau = (12345678)$ , and  $\sigma = (145)(2)(3)(68)(7)$ . The partition  $\pi'$  is represented with dotted lines. One has  $\sigma^{-1}\tau = (123)(4)(58)(67)$ .

### 3. JUCYS-MURPHY ELEMENTS AND YOUNG DIAGRAMS

#### 3.1. Measures Associated with Continuous Young Diagrams

Let  $\omega \in \mathcal{CY}$ , then there exists a unique probability measure  $m_\omega$  with compact support on  $\mathbb{R}$ , determined by

$$G_\omega(z) = G_{m_\omega}(z) = \int_{\mathbb{R}} \frac{1}{z-x} m_\omega(dx)$$

see [AD], [Ke1]. The measure  $m_\omega$  has mean zero, it is called the transition measure of the diagram in [Ke1], [Ke2]. Conversely, given a probability measure  $m$ , with compact support and mean zero, there exists a unique continuous diagram  $\omega$  such that  $m = m_\omega$ . Let us endow, for  $c > 0$ , the space  $\mathcal{CY}_c$  of continuous diagrams, satisfying  $\omega(x) = |x|$  for  $|x| \geq c$ , with the topology of uniform convergence. This is a compact space, and the map  $\omega \mapsto m_\omega$  is a homeomorphism with its image, which is the space of all probability measures with mean zero, and with support in the interval  $[-c, +c]$ , endowed with the topology of weak convergence (see Theorem 2.3 in [Ke1]). Note that the smallest interval  $[-c, +c]$  supporting the measure  $m_\lambda$  is obtained for  $c = \text{width}(\lambda)$ .

Since the generating function of a diagram  $\omega$  coincides with the Cauchy transform of  $m_\omega$ , the free cumulants of the diagram and of the measure also coincide. With these results, we can now prove Theorem 1.2, namely, the operation  $\boxplus$  on continuous diagrams corresponds to the free convolution of measures through the map  $\omega \mapsto m_\omega$ , while the map  $\pi_t$  corresponds to the compression by a free projection of trace  $t$ , as follows easily from Section 2.3.

Since convergence of measures with support in a fixed interval, in the weak topology, is equivalent to the convergence of their moments, we can rephrase Theorems 1.4.1, 1.5.1, and 1.6.1 in terms of the measures associated with the diagrams. In fact it is in this form that we shall prove these theorems. The statements involving measures that we shall prove are the following.

**THEOREM 3.1.1.** *For every  $A > 1$ , and  $n$  positive integer, there exists  $K$ ,  $C_0$ , and  $q_0$  such that for all  $q \geq q_0$ , all  $C > C_0$ , and all  $A$ -balanced Young*

diagrams  $\lambda, \mu \in \mathcal{Y}_q$ , the subspace of the representation  $[\lambda] \otimes [\mu]$ , which is spanned by all the isotypic components  $[v]$  satisfying

$$\left| \int_{\mathbb{R}} x^k m_v(x) - \int_{\mathbb{R}} x^k m_{\Omega_q(x)} \right| \leq Cq^{k/2-1/2} \quad \text{for all } k \leq n,$$

has dimension larger than  $(1 - K/C^2) \dim([\lambda] \otimes [\mu])$ .

**THEOREM 3.1.2.** For every  $A > 1$ ,  $\alpha \in ]0, 1/2]$ , and  $n$  positive integer, there exists  $q_0, K, C_0 > 0$ , such that for all  $q \geq p \geq q_0$ , with  $\alpha q \leq p \leq (1 - \alpha)q$ , all  $C > C_0$ , and all  $A$ -balanced Young diagrams  $\lambda \in \mathcal{Y}_q$ , the subspace of the representation  $[\lambda] \downarrow S_p \times S_{q-p}$ , which is spanned by all the isotypic components  $[v_1] \# [v_2]$  satisfying

$$\left| \int_{\mathbb{R}} x^k m_{v_1}(dx) - \int_{\mathbb{R}} x^k \pi_{(p+1)/(q+1)}(m_\lambda)(dx) \right| \leq Cq^{k/2-1/2} \quad \text{and}$$

$$\left| \int_{\mathbb{R}} x^k m_{v_2}(dx) - \int_{\mathbb{R}} x^k \pi_{(q-p+1)/(q+1)}(m_\lambda)(dx) \right| \leq Cq^{k/2-1/2} \quad \text{for all } k \leq n,$$

has dimension larger than  $(1 - K/C^2) \dim([\lambda])$ .

**THEOREM 3.1.3.** For every  $A > 1$ , and  $n$  positive integer, there exists  $q_0$  and  $K, C_0 > 0$  such that for all  $q \geq q_0$ , all  $C > C_0$ , and all Young diagrams  $\lambda, \mu$  satisfying  $|\lambda|, |\mu| \geq q_0$  and  $\text{width}(\lambda), \text{width}(\mu) \leq A \sqrt{q}$ , the subspace of the representation  $[\lambda] \circ [\mu]$ , which is spanned by all the isotypic components  $[v]$  satisfying

$$\left| \int_{\mathbb{R}} x^k m_v(dx) - \int_{\mathbb{R}} x^k m_\lambda \boxplus m_\mu(dx) \right| \leq Cq^{k/2-1/4} \quad \text{for all } k \leq n,$$

has dimension larger than  $(1 - K/C^2) \dim([\lambda] \circ [\mu])$ .

In order to prove 1.4.1, 1.5.1 and 1.6.1, from 3.1.1, 3.1.2 and 3.1.3, we use the fact that weak convergence of measures with support in  $[-A, A]$  implies uniform convergence of the associated diagrams in  $\mathcal{CY}_A$ , and apply this to the rescaled diagrams  $q^{-1/2}\lambda(q^{1/2} \cdot)$ ,  $q^{-1/2}\mu(q^{1/2} \cdot)$ ..., appearing in the theorems. If we have a good estimate of the modulus of continuity of the inverse map of  $\omega \mapsto m_\omega$ , then we can get a more refined version of Theorems 1.4.1, 1.5.1 and 1.6.1, as explained in the remarks in Section 1.7.

### 3.2. Examples

The square diagram of side  $L$  considered in Section 1.2, corresponds to the symmetric Bernoulli distribution  $\frac{1}{2}(\delta_L + \delta_{-L})$ , while the flat diagram corresponds to arcsine distribution  $(1/\pi)(L^2 - x^2)^{-1/2} dx$  on  $[-L, +L]$ .

The diagram  $\Omega_s$  correspond to the semi-circle distribution of variance  $s$ , namely the distribution with density  $(1/2\pi s)\sqrt{4s-x^2}$  on  $[-2\sqrt{s}, 2\sqrt{s}]$  (see [Ke1]).

### 3.3. Jucys-Murphy Elements and Spectral Interpretation of the Measure Associated with a Young Diagram

For a Young diagram  $\lambda$ , with associated interlacing sequences  $(x_i)_{1 \leq i \leq n}$  and  $(y_i)_{1 \leq i \leq n-1}$ , one has

$$m_\lambda = \sum_{k=1}^n \mu_k \delta_{x_k}$$

where  $\delta_x$  is the Dirac measure at  $x$ , and

$$\mu_k = \frac{\prod_{i=1}^{n-1} (x_k - y_i)}{\prod_{i \neq k} (x_k - x_i)}$$

The interpretation of this measure in terms of representations of the symmetric group is the following. Consider the representation of  $S_{q+1}$  induced by the representation  $[\lambda]$  of  $S_q$ , by the branching rule (see e.g. [JK]), this representation decomposes into  $S_{q+1}$ -irreducible components corresponding to the Young diagrams obtained by adding a box to  $\lambda$ , each appearing with multiplicity one. It is possible to add a box to  $\lambda$  exactly at each of its local minima  $x_k$ . The measure  $m_\lambda$  puts a mass at each point  $x_k$ , which is proportional to the dimension of the representation of  $S_{q+1}$  corresponding to the diagram augmented by a box at  $x_k$ . This follows from the hook formula for the dimension of representations, see [Ke2].

We shall now give a spectral interpretation of the measure  $m_\lambda$ . Recall that the Jucys-Murphy element  $\mathcal{M}_q$  is the sum of transpositions

$$(1q) + (2q) + \dots + (q-1q)$$

in the algebra of the symmetric group  $S_q$ . It is easy to see that  $\mathcal{M}_q$  commutes with  $S_{q-1}$ . The image of  $\mathcal{M}_q$  by an irreducible representation  $[\lambda]$  has a spectrum which can be described as follows. By the branching rule again, the representation  $[\lambda]$ , when restricted to  $S_{q-1}$ , splits into irreducible components corresponding to Young diagrams obtained from  $\lambda$  by taking away one box, each of these diagrams having multiplicity one. By Schur's Lemma, each of these subspaces is an eigenspace of  $[\lambda](\mathcal{M}_q)$ , and one can prove that the corresponding eigenvalue is the abscissa of the lower corner of the box deleted from  $\lambda$ , i.e. one of the numbers  $y_k$ . For a proof of this, see e.g. [OV].

Let  $V_\lambda$  be the space of the representation  $[\lambda]$ , we endow  $End(V_\lambda)$  with its normalized trace  $tr_\lambda$ , and the  $W^*$ -algebra structure such that  $[\lambda]$  is a unitary representation, and we denote by  $\langle \cdot \rangle$  the normalized trace on  $M_{q+1}(\mathbb{C})$ , with its canonical  $W^*$ -algebra structure. We can now prove

**PROPOSITION 3.3.** *The measure  $m_\lambda$  is the distribution of the selfadjoint element*

$$T(\lambda) = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & [\lambda](12) & [\lambda](13) & \dots & [\lambda](1q-1) & [\lambda](1q) \\ 1 & [\lambda](12) & 0 & [\lambda](23) & \dots & [\lambda](2q-1) & [\lambda](2q) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & [\lambda](1q) & [\lambda](2q) & [\lambda](3q) & \dots & [\lambda](q-1q) & 0 \end{pmatrix}$$

in the non-commutative probability space  $(End(V_\lambda) \otimes M_{q+1}(\mathbb{C}), tr_\lambda \otimes \langle \cdot \rangle)$ .

*Proof.* Consider the left regular representation of  $S_{q+1}$ . We identify  $S_{q+1}$  with  $S_q \times \{e; (1\ q+1); (2\ q+1); \dots; (q\ q+1)\}$  by the map  $(\sigma, \tau) \mapsto \sigma\tau$ . In this way we can represent an operator on  $l^2(S_{q+1})$  by a  $(q+1) \times (q+1)$  matrix of operators on  $l^2(S_q)$ . Let  $\sigma \in S_q$  and  $j \in \{1, \dots, q\}$ , then the Jucys–Murphy element acts by the left regular representation as

$$\begin{aligned} \mathcal{M}_{q+1}(\sigma(j\ q+1)) &= \sum_{i=1}^q (i\ q+1) \sigma(j\ q+1) \\ &= \sum_{i=1}^q \sigma(\sigma^{-1}(i)\ q+1)(j\ q+1) \\ &= \sum_{i=1}^q \sigma(i\ q+1)(j\ q+1) \\ &= \sigma + \sum_{\substack{1 \leq i \leq q \\ i \neq j}} \sigma(ji)(i\ q+1) \end{aligned}$$

similarly,

$$\mathcal{M}_{q+1}(\sigma) = \sum_{i=1}^q \sigma(i\ q+1)$$

so that the matrix of  $\mathcal{M}_{q+1}$  acting by the left regular representation is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & (12) & (13) & \cdots & (1q-1) & (1q) \\ 1 & (12) & 0 & (23) & \cdots & (2q-1) & (2q) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (1q) & (2q) & (3q) & \cdots & (q-1q) & 0 \end{pmatrix}$$

where  $(ij)$  is acting by the right regular representation of  $S_q$ . Let us now consider the Young projector  $\pi_\lambda$ , in the left representation of  $S_q$  on  $l^2(S_{q+1})$ . This projector commutes with  $\mathcal{M}_{q+1}$ , and with the action of  $S_q \times S_{q+1}$ , where  $S_q$  acts on the left and  $S_{q+1}$  on the right. Recall that  $l^2(S_{q+1}) = \bigoplus_{A \in \mathcal{A}_{q+1}} [A] \# [A]$  as an  $S_{q+1} \times S_{q+1}$ -module, so that applying the branching rule, the range of  $\pi_\lambda$  reduces, as an  $S_q \times S_{q+1}$ -module, to  $\bigoplus_{k=1}^n [\lambda] \# [A_k]$ , where the sum is over all diagrams  $A_k$  obtained by adding a box to  $\lambda$ . Let  $x_k$  be the value of the abscissa of the lower corner of the box added to  $\lambda$  to form  $A_k$ , then by what we saw before, the Jucys–Murphy element has eigenvalue  $x_k$  on the component of type  $[\lambda] \# [A_k]$ , thus the distribution of  $\pi_\lambda \mathcal{M}_{q+1}$  on the range of  $\pi_\lambda$ , is  $m_\lambda$  (for the reduced trace defined by  $\text{tr}(\pi_\lambda \cdot) / \text{tr}(\pi_\lambda)$ ). On the other hand, its matrix on the range of  $\pi_\lambda$  is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \pi_\lambda(12) & \pi_\lambda(13) & \cdots & \pi_\lambda(1q-1) & \pi_\lambda(1q) \\ 1 & \pi_\lambda(12) & 0 & \pi_\lambda(23) & \cdots & \pi_\lambda(2q-1) & \pi_\lambda(2q) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \pi_\lambda(1q) & \pi_\lambda(2q) & \pi_\lambda(3q) & \cdots & \pi_\lambda(q-1q) & 0 \end{pmatrix}$$

Since  $\pi_\lambda$  is also a Young projector for the right regular representation, one has  $\pi_\lambda \mathcal{M}_{q+1} = \text{Id} \otimes \Gamma(\lambda)$  on the range of  $\pi_\lambda$ , which is  $([\lambda] \# [\lambda]) \otimes \mathbb{C}^{q+1}$  as an  $S_q \times S_q$ -module; we deduce the claim from this. ■

#### 4. ASYMPTOTICS OF CHARACTERS

In this section we shall prove Theorem 1.3. We use again the notations of Section 3.3, in particular  $\Gamma(\lambda)$  denotes the operator defined in Proposition 3.3.

4.1. *Preliminary Estimates*

Let  $\lambda \in \mathcal{Y}_q$  be such that  $\text{width}(\lambda) \leq A \sqrt{q}$ , then one has

$$\left| \int_{\mathbb{R}} x^k m_\lambda(dx) \right| = |\text{tr}_\lambda(\langle \Gamma(\lambda)^k \rangle)| \leq \|\Gamma(\lambda)\|^k \leq A^k q^{k/2}$$

Therefore, the free cumulants of the diagram  $\lambda$  satisfy the estimate  $|C_n(\lambda)| \leq 4^{2n} A^n q^{n/2}$ .

4.2. *New Balanceness Condition*

We shall define, for technical reasons, a refined notion of balanceness for diagrams.

**DEFINITION 4.2.** Let  $A > 1$  and  $n$  be a positive integer, we say that a Young diagram  $\lambda \in \mathcal{Y}_q$  is  $(A, n)$ -balanced if its associated measure  $m_\lambda$  satisfies

$$\left| \int_{\mathbb{R}} x^k m_\lambda(dx) \right| \leq A^k q^{k/2} \quad \text{for all integers } 0 \leq k \leq n.$$

By 4.1, we see that an  $A$ -balanced diagram is  $(A, n)$ -balanced for all  $n$ . Conversely, if  $\lambda$  is  $(A, n)$ -balanced for all  $n$ , then since

$$\text{width}(\lambda) = \|\Gamma(\lambda)\| = \lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}} x^{2n} m_\lambda(dx) \right)^{1/2n} \leq A \sqrt{q},$$

the diagram  $\lambda$  is  $A$ -balanced.

**PROPOSITION 4.2.** For all  $A > 1$  and  $n \geq 2$  integer, there exist a constant  $K > 0$  such that, for all  $(A, n)$ -balanced Young diagrams  $\lambda$ , and all permutations  $\sigma \in S_{|\lambda|}$  satisfying  $|\sigma| \leq n - 2$ , one has

$$\left| \chi_\lambda(\sigma) - \prod_{c|\sigma} |\lambda|^{-|c|-1} C_{|c|+2}(\lambda) \right| \leq K |\lambda|^{-1-|\sigma|/2}$$

where the product is over the disjoint cycles  $c$  of the permutation  $\sigma$ .

We shall prove Proposition 4.2 in the next sections. By the remark following Definition 4.2, we see that Proposition 4.2 implies Theorem 1.3.

### 4.3. Preliminary Combinatorial Lemmas

Let us compute the  $n$ th moment of the distribution of  $\Gamma(\lambda)$ . We make the convention that  $(ij)$  denotes the identity permutation if either  $i$  or  $j$  is equal to zero, then we have

$$\mathrm{tr}_\lambda(\langle \Gamma(\lambda)^n \rangle) = \frac{1}{q+1} \sum_{0 \leq i_1 \neq i_2 \neq \dots \neq i_n \neq i_1 \leq q} \chi_\lambda((i_1 i_2)(i_2 i_3) \cdots (i_n i_1))$$

We shall decompose the set of  $n$ -tuples  $(i_1, \dots, i_n)$  occurring in the above sum according to the set  $J$  of places  $r_1 < r_2 < \dots < r_k$  such that  $i_{r_j} = 0$ . For each  $J \subset \{1, \dots, n\}$ , and  $i_1, \dots, i_n$  such that  $J = \{l \mid i_l = 0\}$ , let  $\pi$  be the partition of  $\{1, \dots, n\} \setminus J$  induced by  $i_1, \dots, i_n$ , namely  $j$  and  $k$  belong to the same component of  $\pi$  if and only if  $i_j = i_k \neq 0$ . Clearly, the conjugacy class of  $(i_1 i_2)(i_2 i_3) \cdots (i_n i_1)$  in  $S_q$  depends only on  $J$  and  $\pi$ . We shall denote by  $h(\pi)$  this conjugacy class, and by  $|h(\pi)|$  the length of any permutation belonging to it. One has thus

$$\mathrm{tr}_\lambda(\langle \Gamma(\lambda)^n \rangle) = \frac{1}{q+1} \sum_{J \subset \{1, \dots, n\}} \sum_{\pi \in \mathcal{P}_a(J, n)} (q)_{\mathrm{comp}(\pi)} \chi_\lambda(h(\pi))$$

where  $\mathcal{P}_a(J, n)$  is the set of *admissible* partitions  $\pi$  of  $\{1, \dots, n\} \setminus J$ , i.e. such that  $j$  and  $j+1 \pmod{n}$  never belong to the same component of  $\pi$ , we denote by  $\mathrm{comp}(\pi)$  the number of components of  $\pi$ , and  $(q)_k = q(q-1)(q-2) \cdots (q-k+1)$ . Note that the sum is over subsets  $J$  not containing two successive elements.

LEMMA 4.3.1. *Let  $\pi \in \mathcal{P}_a(\emptyset, n)$  be a partition with a crossing, then*

$$|h(\pi)| + n \geq 2 \mathrm{comp}(\pi).$$

*Proof.* We choose  $q > n$ . Let  $i_1, \dots, i_n$  induce an admissible partition  $\pi$ , with at most  $n-1$  components, and let  $k \in \{1, \dots, n\}$  belong to some component of  $\pi$  with at least two elements. Let  $i'_k \in \{1, \dots, q\}$  be distinct from all  $i_1, i_2, \dots, i_n$ , and let  $\pi'$  be the admissible partition induced by  $i_1, \dots, i_{k-1}, i'_k, i_{k+1}, \dots, i_n$ , then  $\mathrm{comp}(\pi') = \mathrm{comp}(\pi) + 1$ . Write

$$\begin{aligned} \sigma &= (i_1 i_2)(i_2 i_3) \cdots (i_{k-1} i_k)(i_k i_{k+1}) \cdots (i_n i_1) \\ &= \sigma_1(i_{k-1} i_k)(i_k i_{k+1}) \sigma_2 \end{aligned}$$

and

$$\begin{aligned} \sigma' &= (i_1 i_2)(i_2 i_3) \cdots (i_{k-1} i'_k)(i'_k i_{k+1}) \cdots (i_n i_1) \\ &= \sigma_1(i_{k-1} i'_k)(i'_k i_{k+1}) \sigma_2, \end{aligned}$$

then  $\sigma'\sigma^{-1} = \sigma_1(i_{k-1}i'_k)(i'_ki_{k+1})(i_{k+1}i_k)(i_ki_{k-1})\sigma_1^{-1} = \sigma_1(i_ki_{k+1}i'_k)\sigma_1^{-1}$  if  $i_{k-1} \neq i_{k+1}$  and  $\sigma'\sigma^{-1} = e$  if  $i_{k-1} = i_{k+1}$ , hence  $|\sigma'\sigma^{-1}| \leq 2$ . The class of  $\sigma$  is  $h(\pi)$ , while the class of  $\sigma'$  is  $h(\pi')$ , thus  $|h(\pi')| \leq |h(\pi)| + 2$ , and it follows that the quantity  $|h(\pi)| + n - 2 \text{comp}(\pi)$  does not increase when we pass from the partition  $\pi$  to  $\pi'$ . If  $\pi$  has a crossing, then there exists  $j < k < l < m$  with  $j \sim l$  and  $k \sim m$  for  $\pi$ , thus if we consider the partition  $\pi''$  consisting of two pairs  $\{j, l\}$ ,  $\{k, m\}$  and  $n - 4$  singletons, then we can pass from  $\pi$  to  $\pi''$  through a sequence of changes of the form  $\pi \mapsto \pi'$  as above thus, by what we have just seen, one has

$$|h(\pi)| + n - 2 \text{comp}(\pi) \geq |h(\pi'')| + n - 2 \text{comp}(\pi''),$$

but we can compute explicitly that  $|h(\pi'')| + n = 2 \text{comp}(\pi'') = 2n - 4$ , this proves the lemma. ■

LEMMA 4.3.2. *Let  $J$  be nonempty, and let  $\pi \in \mathcal{P}_a(J, n)$ , then*

$$|h(\pi)| + n \geq 2 \text{comp}(\pi).$$

*Proof.* We start with  $\pi$  and  $k$ , such that  $i_k \neq 0$ , and  $k$  belongs to a class of  $\pi$  with at least two elements, as in the proof of the preceding lemma. We build the new partition  $\pi'$  and the permutation  $\sigma'$ , then we can check that even if  $i_{k-1}$  or  $i_{k+1}$  is zero, we still have  $|\sigma'\sigma^{-1}| \leq 2$ , and thus, proceeding as in Lemma 4.3.1, we can reduce the claim to the case where all  $i_k$ , for  $k \in \{1, \dots, n\} \setminus J$  are distinct. In this case, by a direct computation, one checks, denoting by  $|J|$  the number of elements of  $J$ , that

$$|h(\pi)| = n - |J| - 1 = \text{comp}(\pi) - 1,$$

and the result follows. ■

LEMMA 4.3.3. *Let  $\pi \in \mathcal{P}_a(\emptyset, n) \cap NC(n)$ , then one has*

$$2 + |h(\pi)| + n = 2 \text{comp}(\pi).$$

*Furthermore, if  $2^{k_2}3^{k_3}4^{k_4} \dots$  is the conjugacy class of  $t(K(\pi))$ , then the conjugacy class of  $h(\pi)$  is  $2^{k_3}3^{k_4}4^{k_5} \dots$ .*

*Proof.* The claim is easy to check when  $n = 2$ , or when  $\pi$  has  $n$  components. We prove the result by induction on  $n$ . Suppose that it is proved up to  $n - 1$ , and consider a non-crossing admissible partition of  $\{1, \dots, n\}$ , with strictly less than  $n$  components. Then there exists some  $j, k$  such that  $j < k - 1$ ,  $j \sim k$  and the classes of  $j + 1, j + 2, \dots, k - 1$  consist of singletons.

Then for any  $i_1, \dots, i_n$  inducing  $\pi$ , one has  $i_j = i_k$ , and  $(i_j i_{j+1}) \cdots (i_{k-1} i_k) = (i_{j+1} i_{j+2} \cdots i_{k-1})$ , therefore

$$(i_1 i_2)(i_2 i_3) \cdots (i_n i_1) \\ = [(i_1 i_2) \cdots (i_{j-1} i_j)(i_j i_{k+1}) \cdots (i_n i_1)] \times (i_{j+1} i_{j+2} \cdots i_{k-1})$$

where  $(i_1 i_2) \cdots (i_{j-1} i_j)(i_j i_{k+1}) \cdots (i_n i_1)$  and  $(i_{j+1} i_{j+2} \cdots i_{k-1})$  have disjoint supports, thus the class  $h(\pi)$  consists of the product of a disjoint cycle of length  $k-j-2$  with the class of  $h(\pi')$  where  $\pi'$  is the non-crossing partition of  $\{1, \dots, j\} \cup \{k+1, \dots, n\}$  induced by  $i_1, \dots, i_j, i_{k+1}, \dots, i_n$ . On the other hand, one sees that the conjugacy class of  $t(K(\pi))$  consists of that of the Kreweras complement of  $\pi'$ , to which is added a disjoint cycle of length  $k-j-1$ . The first claim follows by induction. The fact that  $2 + |h(\pi)| + n = 2 \text{comp}(\pi)$  also follows from these considerations by induction. ■

**LEMMA 4.3.4.** *Let  $\pi \in \mathcal{P}_a(J, n)$ , then  $|h(\pi)| \leq n-2$ , and there is equality if and only if  $J = \emptyset$  and  $\pi$  is the partition with  $n$  components, or  $J$  has one element and  $\pi$  is a partition with  $n-1$  components. In these cases,  $h(\pi)$  is a cycle of order  $n-1$ .*

*Proof.* The number of elements in the support of  $h(\pi)$  is at most equal to the number of components of  $\pi$ . Since  $|\sigma| \leq s(\sigma) - 1$  for any permutation, it follows that  $|h(\pi)| \leq \text{comp}(\pi) - 1 \leq n-1$ . If  $J = \emptyset$ , then  $h(\pi)$  is a product of  $n$  transpositions, hence its length has the same parity as  $n$  (for signature reasons), since it is at most  $n-1$ , it cannot be more than  $n-2$ . If  $J \neq \emptyset$ , then  $h(\pi)$  is a product of at most  $n-2$  transpositions, and thus  $|h(\pi)| \leq n-2$ . If  $|h(\pi)| = n-2$ , then  $\pi$  has at least  $n-1$  components. In the two cases of the lemma one can check the last assertion directly, so that it remains to rule out the possibility that  $J = \emptyset$  and  $\pi$  consists of  $n-2$  singletons and one pair. Assume this is the case, i.e.  $i_r = i_l$  with  $r+1 < l$ , and all other  $i'_k$ s are distinct, then one sees by a direct computation that  $|h(\pi)| = n-4$ . ■

**LEMMA 4.3.5.** *Let  $\sigma_1$  and  $\sigma_2$  be permutations having  $k$  elements in common in their supports ( $k \geq 1$ ), then  $|\sigma_1 \sigma_2| \geq |\sigma_1| + |\sigma_2| + 2 - 2k$ .*

*Proof.* Suppose first that  $k = 1$ , then  $\sigma_1 = \beta_1 \gamma_1$ , and  $\sigma_2 = \beta_2 \gamma_2$  where  $\gamma_1$  and  $\gamma_2$  are the cycles containing the common element. Thus  $\sigma_1 \sigma_2 = \beta_1 \beta_2 (\gamma_1 \gamma_2)$ , but  $\gamma_1 \gamma_2$  is a cycle of length  $|\gamma_1| + |\gamma_2|$ , hence  $|\sigma_1 \sigma_2| = |\beta_1| + |\beta_2| + |\gamma_1 \gamma_2| = |\sigma_1| + |\sigma_2|$ . If  $k > 1$ , let  $a_0, a_1, \dots, a_{k-1}$  be the elements in common. Choose elements  $a'_1, \dots, a'_{k-1}$  which do not belong to the supports

of either  $\sigma_1$  or  $\sigma_2$  (we assume  $q$  is large enough), and let  $\tau = (a_1 a'_1)(a_2 a'_2) \cdots (a_{k-1} a'_{k-1})$ , then  $\tau\sigma_1\tau^{-1}$  and  $\sigma_2$  have only one common element  $a_0$  in their supports, hence

$$\begin{aligned} |\sigma_1| + |\sigma_2| &= |\tau\sigma_1\tau^{-1}| + |\sigma_2| \\ &= |\tau\sigma_1\tau^{-1}\sigma_2| \\ &= |\tau\sigma_1\tau^{-1}\sigma_1^{-1}\sigma_1\sigma_2| \\ &\leq |\tau| + |\sigma_1\tau^{-1}\sigma_1^{-1}| + |\sigma_1\sigma_2| \\ &\leq 2k - 2 + |\sigma_1\sigma_2| \quad \blacksquare \end{aligned}$$

**LEMMA 4.3.6.** *Let  $\sigma_1$  and  $\sigma_2$  be permutations such that  $|\sigma_1 + \sigma_2| = |\sigma_1| + |\sigma_2|$ , then either  $\sigma_1$  and  $\sigma_2$  have disjoint supports, or  $c'(\sigma_1\sigma_2) \leq c'(\sigma_1) + c'(\sigma_2) - 1$ , where  $c'(\sigma)$  denotes the number of non-trivial cycles of the permutation  $\sigma$ .*

*Proof.* For any permutation one has  $|\sigma| = s(\sigma) - c'(\sigma)$ . If  $\sigma_1$  and  $\sigma_2$  do not have disjoint support, then  $s(\sigma_1\sigma_2) \leq s(\sigma_1) + s(\sigma_2) - 1$ . The claim follows easily.  $\blacksquare$

#### 4.4. Induction on the Length of $\sigma$

We shall now prove Proposition 4.2 by induction on  $n$ . The case  $n = 2$  is trivial. Assume Proposition 4.2 holds for  $n - 1$ , (with  $n \geq 3$ ), then it follows that  $|\chi_\lambda(\sigma)| = O(q^{-|\sigma|/2})$  uniformly over all  $(A, n - 1)$ -balanced diagrams  $\lambda$ , and permutations  $\sigma$  of length  $\leq n - 3$ . Recall that from Section 4.3 we have

$$\text{tr}_\lambda(\langle \Gamma(\lambda)^n \rangle) = \frac{1}{q + 1} \sum_{J \subset \{1, \dots, n\}} \sum_{\pi \in \mathcal{P}_q(J, n)} (q)_{\text{comp}(\pi)} \chi_\lambda(h(\pi))$$

By Lemma 4.3.4, all the terms  $h(\pi)$  in this expression have length at most  $n - 3$ , except the ones where  $J = \emptyset$  and  $\pi$  is the partition with  $n$  components, or  $J$  has one element, and  $\pi$  is a partition with  $n - 1$  components. These last terms give a contribution of

$$\begin{aligned} &\frac{1}{q + 1} ((q)_n + n(q)_{n-1}) \chi_\lambda((12 \cdots n - 1)) \\ &= q^{n-1}(1 + O(q^{-1})) \chi_\lambda((12 \cdots n - 1)) \end{aligned}$$

We now evaluate the contribution of other partitions. We use the fact that for  $\pi$  not in the preceding set, by induction hypothesis,  $\chi_\lambda(h(\pi)) = O(q^{-|h(\pi)|/2})$ , uniformly over  $(A, n)$ -balanced diagrams (in the

following, we shall not repeat each time the words “uniformly over  $(A, n)$ -balanced diagrams”, which will be implicit). If  $\pi \in \mathcal{P}_a(\emptyset, n)$  is a partition with a crossing, or if  $J$  has at least one element, and  $\pi \in \mathcal{P}_a(J, n)$ , then from Lemmas 4.3.1 and 4.3.2, one has  $(1/(q+1))(q)_{\text{comp}(\pi)} \chi_\lambda(h(\pi)) = O(q^{n/2-1})$ . Therefore one has

$$\begin{aligned} \text{tr}_\lambda(\langle \Gamma(\lambda)^n \rangle) &= q^{n-1}(1 + O(q^{-1})) \chi_\lambda((12 \cdots n-1)) \\ &\quad + \frac{1}{q+1} \sum_{\pi \in \mathcal{P}_a(\emptyset, n) \cap NC(n) \setminus \{0_n\}} (q)_{\text{comp}(\pi)} \chi_\lambda(h(\pi)) + O(q^{n/2-1}) \end{aligned}$$

where  $0_n$  is the partition with  $n$  classes.

The set of Kreweras complements of partitions  $\pi \in \mathcal{P}_a(\emptyset, n) \cap NC(n)$  is the set of non-crossing partitions with no singleton, which we denote by  $NC_1(n)$ . From Lemma 4.3.3 and the induction hypothesis we deduce that

$$\begin{aligned} \text{tr}_\lambda(\langle \Gamma(\lambda)^n \rangle) &= q^{n-1}(1 + O(q^{-1})) \chi_\lambda((12 \cdots n-1)) \\ &\quad + \sum_{\pi \in NC_1(n) \setminus \{0_n\}} \prod_{V \in \pi} C_{|V|}(\lambda) + O(q^{n/2-1}) \end{aligned}$$

We compare this expression with the expression for the moments of the distribution of  $\Gamma(\lambda)$  in terms of its cumulants (see Section 2.4). Since the first moment of  $\Gamma(\lambda)$  is zero, one has

$$\text{tr}_\lambda(\langle \Gamma(\lambda)^n \rangle) = \sum_{\pi \in NC_1(n)} \prod_{V \in \pi} C_{|V|}(\lambda).$$

Comparing these two relations, we have

$$\chi_\lambda((12 \cdots n-1)) = q^{-n+1} C_n(\lambda) + O(q^{-n/2}),$$

uniformly over all  $(A, n)$ -balanced diagrams, and the induction hypothesis is verified for cycles of length  $n-2$ .

We now continue the proof by induction on the number of non-trivial cycles of the permutation  $\sigma$ . So we assume that Proposition 4.2 holds for permutations of length  $n-2$ , with at most  $r-1$  non-trivial cycles, with  $r \geq 2$ . Let now  $\sigma$  be a permutation of length  $n-2$ , with  $r$  non-trivial cycles, and write it as  $\sigma = \sigma_1 \sigma_2$ , a non-trivial product of two permutations with disjoint supports. We denote by  $\text{card}(\bar{\sigma}_i)$ , for  $i=1, 2$ , the number of all elements conjugate to  $\sigma_i$  in  $S_q$ , and by  $\Sigma_i$  the sum of these elements in the group algebra. Let  $\lambda \in \mathcal{Y}_q$ , then  $[\lambda](\Sigma_i)$ ;  $i=1, 2$  are multiples of the identity, and thus

$$\begin{aligned} \text{tr}_\lambda([\lambda](\Sigma_1 \Sigma_2)) &= \text{tr}_\lambda([\lambda](\Sigma_1)) \text{tr}_\lambda([\lambda](\Sigma_2)) \\ &= \text{card}(\bar{\sigma}_1) \text{card}(\bar{\sigma}_2) \chi_\lambda(\sigma_1) \chi_\lambda(\sigma_2). \end{aligned}$$

One has  $\text{card}(\bar{\sigma}_i) = C_i q^{s(\sigma_i)} + O(q^{s(\sigma_i)-1})$ , where  $C_i$  are positive combinatorial terms, independent of  $q$  (they depend only on the numbers  $k_2, k_3, \dots$  of cycles of lengths  $\geq 2$  of  $\sigma_i$ ). Let us expand the product  $\Sigma_1 \Sigma_2$ . Since  $s(\sigma_1 \sigma_2) = s(\sigma_1) + s(\sigma_2)$ , we find that the terms which are conjugate to  $\sigma_1 \sigma_2$ , are obtained from pairs of permutations  $(\sigma'_1, \sigma'_2)$ , having disjoint supports, in the respective conjugacy classes  $\bar{\sigma}_1, \bar{\sigma}_2$ . Their number is  $C_1 C_2 q^{s(\sigma_1) + s(\sigma_2)} + O(q^{s(\sigma_1) + s(\sigma_2) - 1})$ . The terms which are not conjugate to  $\sigma_1 \sigma_2$  correspond to pairs  $(\sigma'_1, \sigma'_2)$  which have some element in common in their supports. The number of such pairs having  $k \geq 1$  elements in common is of the order  $O(q^{s(\sigma_1) + s(\sigma_2) - k})$ . By Lemma 4.3.5, they satisfy  $|\sigma'_1 \sigma'_2| \geq |\sigma_1 \sigma_2| + 2 - 2k$ , and by Lemma 4.3.6, if  $|\sigma'_1 \sigma'_2| = |\sigma_1 \sigma_2| = |\sigma_1| + |\sigma_2|$ , then the number of nontrivial cycles of  $\sigma'_1 \sigma'_2$  is at most  $r - 1$ , so that the induction hypothesis gives us the estimate  $\chi_\lambda(\sigma'_1 \sigma'_2) = O(q^{-|\sigma_1 \sigma_2|/2})$ . Thus, these terms give a contribution to  $\text{tr}_\lambda([\lambda](\Sigma_1 \Sigma_2))$ , which is of order

$$O(q^{s(\sigma_1) + s(\sigma_2) - k - |\sigma_1|/2 - |\sigma_2|/2 - 1 + k}) = O(q^{s(\sigma_1) + s(\sigma_2) - |\sigma_1|/2 - |\sigma_2|/2 - 1}).$$

This implies that

$$\begin{aligned} & C_1 C_2 q^{s(\sigma_1) + s(\sigma_2)} \chi_\lambda(\sigma_1) \chi_\lambda(\sigma_2) \\ &= C_1 C_2 q^{s(\sigma_1) + s(\sigma_2)} \chi_\lambda(\sigma_1 \sigma_2) + O(q^{s(\sigma_1) + s(\sigma_2) - |\sigma_1|/2 - |\sigma_2|/2 - 1}) \end{aligned}$$

so that one has

$$\chi_\lambda(\sigma) = \chi_\lambda(\sigma_1) \chi_\lambda(\sigma_2) + O(q^{-|\sigma_1|/2 - |\sigma_2|/2 - 1})$$

uniformly over all  $(A, n)$ -balanced diagrams. Using the induction hypothesis for  $\sigma_1$  and  $\sigma_2$ , we have the correct estimate for  $\sigma$ .

This proves Proposition 4.2. ■

Corollary 1.3 follows from the preceding proof, in fact one sees that there is a version of that corollary with  $(A, n)$ -balanced diagrams.

## 5. TENSOR PRODUCTS

### 5.1. Reduction to Markov's Inequality

We use again the notations of Section 3.3 and Section 4. Let now  $\xi$  be a finite dimensional, reducible representation of  $S_q$ , on some space  $V_\xi$ , then one can define an operator  $\Gamma(\xi)$  in  $\text{End}(V_\xi) \otimes M_{q+1}(\mathbb{C})$  by a formula similar to the definition of  $\Gamma(\lambda)$  in Proposition 3.3. We denote by  $\text{tr}_\xi$  the normalized trace on  $\text{End}(V_\xi)$ .

LEMMA 5.1.1. *Let  $A > 1$ , then for all  $n, m$  positive integers, one has*

$$\text{tr}_{[\lambda] \otimes [\mu]}(\langle \Gamma([\lambda] \otimes [\mu])^n \rangle) = \int_{\mathbb{R}} x^n m_{\Omega_q}(dx) + O(q^{n/2-1}) \quad (5.1.1.)$$

$$\begin{aligned} & \text{tr}_{[\lambda] \otimes [\mu]}(\langle \Gamma([\lambda] \otimes [\mu])^n \rangle \langle \Gamma([\lambda] \otimes [\mu])^m \rangle) \\ &= \text{tr}_{[\lambda] \otimes [\mu]}(\langle \Gamma([\lambda] \otimes [\mu])^n \rangle) \text{tr}_{[\lambda] \otimes [\mu]}(\langle \Gamma([\lambda] \otimes [\mu])^m \rangle) \\ & \quad + O(q^{n/2+m/2-1}) \end{aligned} \quad (5.1.2)$$

uniformly over all  $A$ -balanced Young diagrams  $\lambda, \mu \in \mathcal{Y}_q$ .

*Proof.* By the same computation as in Section 4.3, since the normalized character of  $[\lambda] \otimes [\mu]$  is  $\chi_\lambda \chi_\mu$ , one has

$$\begin{aligned} & \text{tr}_{[\lambda] \otimes [\mu]}(\langle \Gamma([\lambda] \otimes [\mu])^n \rangle) \\ &= \frac{1}{q+1} \sum_{J=\{1, \dots, n\}} \sum_{\pi \in \mathcal{P}_a(J, n)} (q)_{\text{comp}(\pi)} \chi_\lambda \chi_\mu(h(\pi)) \end{aligned}$$

By Corollary 1.3 we see that for a typical term in the sum one has

$$\frac{1}{q+1} (q)_{\text{comp}(\pi)} \chi_\lambda \chi_\mu(h(\pi)) = O(q^{-1 + \text{comp}(\pi) - |h(\pi)|}).$$

By Lemmas 4.3.1 and 4.3.2 and 4.3.3, for all  $\pi$ , one has  $2 + |h(\pi)| + n \geq 2 \text{comp}(\pi)$ , so that the only terms which are not of order  $O(q^{n/2-1})$ , satisfy  $-1 + \text{comp}(\pi) - |h(\pi)|/2 = n/2$  and  $|h(\pi)| = 0$ , i.e.  $\pi \in \mathcal{P}_a(\emptyset, n)$  is non-crossing and  $h(\pi) = e$ . By Lemma 4.3.3 there are no such partitions if  $n$  is odd and for even  $n$ , the partitions satisfying these requirements are the non-crossing partitions whose Kreweras complement is a partition of  $\{1, \dots, n\}$  into  $n/2$  pairs. It is well known that the number of such partitions is equal to the Catalan number  $(2p)!/p!(p+1)!$ , where  $n = 2p$ . Each such partition gives a contribution of  $q^p + O(q^{p-1})$  to the sum. On the other hand, one has  $\int_{\mathbb{R}} x^n m_{\Omega_q}(dx) = 0$  if  $n$  is odd,  $\int_{\mathbb{R}} x^{2p} m_{\Omega_q}(dx) = q^p((2p)!/p!(p+1)!)$ . This proves part (5.1.1) of the estimates.

In order to prove (5.1.2) we expand

$$\begin{aligned} & \text{tr}_{[\lambda] \otimes [\mu]}(\langle \Gamma([\lambda] \otimes [\mu])^n \rangle \langle \Gamma([\lambda] \otimes [\mu])^m \rangle) \\ &= \frac{1}{(q+1)^2} \sum_{\substack{0 \leq i_1 \neq i_2 \neq \dots \neq i_n \neq i_1 \leq q \\ 0 \leq i_{n+1} \neq i_{n+2} \neq \dots \neq i_{n+m} \neq i_{n+1} \leq q}} \chi_\lambda \chi_\mu((i_1 i_2) \cdots (i_n i_1)(i_{n+1} i_{n+2}) \cdots (i_{n+m} i_{n+1})) \end{aligned}$$

We write this sum as a sum over sets  $J$  and partitions of  $\{1, \dots, n\} \cup \{n+1, \dots, n+m\} \setminus J$ , denoting  $\tilde{h}(\pi)$  the conjugacy class associated with a partition, we get

$$\begin{aligned} & \text{tr}_{[\lambda] \otimes [\mu]}(\langle \Gamma([\lambda] \otimes [\mu])^n \rangle \langle \Gamma([\lambda] \otimes [\mu])^n \rangle) \\ &= \frac{1}{(q+1)^2} \sum_{J \subset \{1, \dots, n+m\}} \sum_{\pi \in \tilde{\mathcal{P}}_a(J, \{1, \dots, n+m\})} (q)_{\text{comp}(\pi)} \chi_\lambda \chi_\mu(\tilde{h}(\pi)) \end{aligned}$$

where admissible partitions in  $\tilde{\mathcal{P}}_a(J, \{1, \dots, n+m\})$  are partitions of  $\{1, \dots, n\} \cup \{n+1, \dots, n+m\} \setminus J$ , such that  $j \not\sim j+1$ , for  $j = 1, \dots, n-1$ ,  $1 \not\sim n$ ,  $n+j \not\sim n+j+1$  for  $j = 1, \dots, m-1$ , and  $n+1 \not\sim n+m$ . We have the analogue of Lemmas 4.3.1 and 4.3.2.

LEMMA 5.1.2. *Let  $\pi$  be an admissible partition, as above, of  $\{1, \dots, n+m\}$  such that there exists  $k \in \{1, \dots, n\}$  and  $l \in \{n+1, \dots, n+m\}$  with  $k \sim l$ , then one has*

$$2 + |\tilde{h}(\pi)| + n + m \geq 2 \text{comp}(\pi).$$

*Proof.* As in the proof of Lemma 4.3.1, we can reduce the claim to the verification in the case where  $\pi$  consists in one pair  $\{k, l\}$ , with  $k \in \{1, \dots, n\}$  and  $l \in \{n+1, \dots, n+m\}$ , and  $n+m-2-|J|$  singletons, which we leave to check to the reader. ■

*End of proof of Lemma 5.1.1.* With the help of Lemmas 5.1.2, we see that the sum of terms corresponding to partitions which connect  $\{1, \dots, n\}$  and  $\{n+1, \dots, n+m\}$  can be neglected, and using Lemma 4.3.2, one can neglect the terms with  $J \neq \emptyset$ , so that

$$\begin{aligned} & \text{tr}_{[\lambda] \otimes [\mu]}(\langle \Gamma([\lambda] \otimes [\mu])^n \rangle \langle \Gamma([\lambda] \otimes [\mu])^m \rangle) \\ &= \frac{1}{(q+1)^2} \sum_{\substack{\pi_1 \in \mathcal{P}_a(\emptyset, \{1, \dots, n\}) \\ \pi_2 \in \mathcal{P}_a(\emptyset, \{n+1, \dots, n+m\})}} (q)_{\text{comp}(\pi_1) + \text{comp}(\pi_2)} \chi_\lambda \chi_\mu(h(\pi_1) h(\pi_2)) + O(q^{n/2+m/2-1}) \end{aligned}$$

where  $h(\pi_1) h(\pi_2)$  denotes the class of the product of two permutations of classes  $h(\pi_1)$  and  $h(\pi_2)$  with disjoint supports. Note that one has  $(q)_{\text{comp}(\pi_1) + \text{comp}(\pi_2)} \sim (q)_{\text{comp}(\pi_1)} (q)_{\text{comp}(\pi_2)}$ . Using now the approximate factorization of characters (i.e. Corollary 1.3) and the estimate  $\chi_\mu(\sigma) \chi_\lambda(\sigma) = O(q^{-|\sigma|})$ , we can replace the terms  $\chi_\mu \chi_\lambda(h(\pi_1) h(\pi_2))$  by  $\chi_\mu \chi_\lambda(h(\pi_1)) \chi_\mu \chi_\lambda(h(\pi_2))$  and we have

$$\begin{aligned} & \text{tr}_{[\lambda] \otimes [\mu]}(\langle \Gamma([\lambda] \otimes [\mu])^n \rangle \langle \Gamma([\lambda] \otimes [\mu])^m \rangle) \\ &= \text{tr}_{[\lambda] \otimes [\mu]}(\langle \Gamma([\lambda] \otimes [\mu])^n \rangle) \text{tr}_{[\lambda] \otimes [\mu]}(\langle \Gamma([\lambda] \otimes [\mu])^m \rangle) \\ & \quad + O(q^{n/2+m/2-1}). \quad \blacksquare \end{aligned}$$

5.2. End of the Proof of Theorem 3.1.1

Let  $\lambda \in \mathcal{Y}_q$ , then the operator  $\langle \Gamma(\lambda)^n \rangle \in \text{End}(V_\lambda)$  commutes with all  $[\lambda](\sigma)$ , for  $\sigma \in S_q$ , so that by Schur’s lemma, it is a multiple of the identity, its eigenvalue being the  $n$ th moment of the distribution of  $\Gamma(\lambda)$ . It follows that for a reducible representation  $\xi$ , the operator  $\langle \Gamma(\xi)^n \rangle$  is a multiple of the identity on each isotypic component of  $\xi$ , with the eigenvalue  $\text{tr}_\lambda(\langle \Gamma(\lambda)^n \rangle)$  on the component of type  $[\lambda]$ . Denote by  $t_{\lambda\mu}^v$  the multiplicity of  $[v]$  in  $[\lambda] \otimes [\mu]$ . From the above considerations we deduce that one has

$$\begin{aligned} & \text{tr}_{[\lambda] \otimes [\mu]} \left( \left( \langle \Gamma([\lambda] \otimes [\mu])^n \rangle - \int_{\mathbb{R}} x^n m_{\Omega_q}(dx) \right)^2 \right) \\ &= \sum_{v \in \mathcal{Y}_q} t_{\lambda\mu}^v \frac{\dim([v])}{\dim([\lambda] \otimes [\mu])} \left( \int_{\mathbb{R}} x^n m_v(dx) - \int_{\mathbb{R}} x^n m_{\Omega_q}(dx) \right)^2. \end{aligned}$$

Using estimates (5.1.1) and (5.1.2), we see that this expression is of order  $O(q^{n-1})$ . Let  $H(C)$  be the set of  $v \in \mathcal{Y}_q$  such that

$$\left| \int_{\mathbb{R}} x^n m_v(dx) - \int_{\mathbb{R}} x^n m_{\Omega_q}(dx) \right| > Cq^{n/2-1/2}$$

Applying Markov’s inequality, we obtain that

$$\sum_{v \in H(C)} t_{\lambda\mu}^v \frac{\dim([v])}{\dim([\lambda] \otimes [\mu])} \leq C^{-2} q^{-n+1} \times O(q^{n-1})$$

uniformly over  $A$ -balanced Young diagrams in  $\mathcal{Y}_q$ . Theorem 3.1.1 follows from this.  $\blacksquare$

5.3. Proof of Theorem 1.4.2

Fix  $A$ ,  $\varepsilon$ ,  $\delta$ , and choose  $C$  large enough in Theorem 3.1.1 so that  $K/C^2 < \delta$ . Then for  $q$  large enough, the diagrams occurring in the “good” set of Theorem 3.1.1 will be  $(A + 1, n)$ -balanced, so that we can apply Proposition 4.2 to them, and deduce that the corresponding characters are given asymptotically by the cumulants of the diagram  $\Omega_q$ . We leave details to the reader.

6. RESTRICTIONS

6.1. *Asymptotic Freeness with Diagonal Operators*

Again we use the same notations as in Section 4. We shall prove an estimate which essentially means that  $\Gamma(\lambda)$  becomes asymptotically free with diagonal operators in  $M_{q+1}(\mathbb{C})$ , when  $q \rightarrow \infty$ . We consider  $D^{(j)}$ , for  $j = 1, \dots, n$ , diagonal matrices in  $M_{q+1}(\mathbb{C})$ , which we identify with their dilations to  $I_{V_\lambda} \otimes D^{(j)} \in \text{End}(V_\lambda) \otimes M_{q+1}(\mathbb{C})$ .

LEMMA 6.1. *Let  $A > 1$  and  $n, m$  be positive integers, denoting by  $R$  the free cumulants associated with the trace  $\text{tr}_\lambda \otimes \langle \cdot \rangle$ , one has*

$$\begin{aligned} & \text{tr}_\lambda(\langle D^{(1)}\Gamma(\lambda) D^{(2)}\Gamma(\lambda) \dots D^{(n)}\Gamma(\lambda) \rangle) \\ &= \sum_{\pi \in \text{NC}(n)} \langle \cdot \rangle[\pi](D^{(1)}, \dots, D^{(n)}) R[K(\pi)](\Gamma(\lambda), \dots, \Gamma(\lambda)) + O(q^{n/2-1}) \end{aligned} \tag{6.1.1}$$

$$\begin{aligned} & \text{tr}_\lambda(\langle D^{(1)}\Gamma(\lambda) D^{(2)} \dots D^{(n)}\Gamma(\lambda) \rangle \langle D^{(n+1)}\Gamma(\lambda) D^{(n+2)} \dots D^{(n+m)}\Gamma(\lambda) \rangle) \\ &= \text{tr}_\lambda(\langle D^{(1)}\Gamma(\lambda) D^{(n+2)} \dots D^{(n)}\Gamma(\lambda) \rangle) \\ & \quad \times \text{tr}_\lambda(\langle D^{(n+1)}\Gamma(\lambda) D^{(n+2)} \dots D^{(n+m)}\Gamma(\lambda) \rangle) \\ & \quad + O(q^{n/2+m/2-1}) \end{aligned} \tag{6.1.2}$$

uniformly over all  $A$ -balanced Young diagrams  $\lambda \in \mathcal{Y}_q$ , and all diagonal matrices  $D^{(1)}, \dots, D^{(n+m)} \in M_{q+1}(\mathbb{C})$  having all their coefficients bounded by 1 in absolute value.

*Proof.* We first expand the left-hand side of (6.1.1); we get

$$\frac{1}{q+1} \sum_{0 \leq i_1 \neq i_2 \neq \dots \neq i_n \neq i_1 \leq q} D_{i_1 i_1}^{(1)} \dots D_{i_n i_n}^{(n)} \chi_\lambda((i_1 i_2)(i_2 i_3) \dots (i_n i_1))$$

As in Section 4.3, we rewrite this in terms of admissible partitions of  $\{1, \dots, n\} \setminus J$ . The term corresponding to a partition  $\pi$  is

$$\frac{1}{q+1} \chi_\lambda(h(\pi)) \sum_{i_1, \dots, i_n \text{ induce } \pi} D_{i_1 i_1}^{(1)} \dots D_{i_n i_n}^{(n)}.$$

Since all coefficients of the matrices  $D^{(k)}$  are bounded by 1, as in Section 4, we see that we can neglect the terms corresponding to  $\pi$  not

in  $\mathcal{P}_a(\emptyset, n) \cap NC(n)$ . On the other hand, for  $\pi \in \mathcal{P}_a(\emptyset, n) \cap NC(n)$  one has

$$\begin{aligned} & \sum_{i_1, \dots, i_n \text{ induce } \pi} D_{i_1 i_1}^{(1)} \cdots D_{i_n i_n}^{(n)} \\ &= q^{\text{comp}(\pi)} \langle \cdot \rangle [\pi](D^{(1)}, \dots, D^{(n)}) + O(q^{\text{comp}(\pi)-1}). \end{aligned}$$

Note that in (6.1.1), the partitions such that  $K(\pi)$  has one-element components do not contribute. By Theorem 1.3, and Lemma 4.3.3, one has

$$R[K(\pi)](\Gamma(\lambda), \dots, \Gamma(\lambda)) = q^{n/2 + |h(\pi)|/2} \chi_\lambda(h(\pi)) + O(q^{n/2-1})$$

for  $\pi \in NC(n)$ . This implies (6.1.1). The estimate (6.1.2) can be proved along the same lines as the estimate (5.1.2) of Section 5.2, using Lemma 5.1.2. We omit the details. ■

## 6.2. End of Proof of Theorem 3.1.2

Let  $p \leq q$ , we shall apply Lemma 6.1 in the case where the matrices  $D^{(k)}$  are all equal to  $\Pi$ , the matrix with one's on the first  $p+1$  diagonal elements, and zeros outside. The matrix  $\Pi \Gamma(\lambda) \Pi$  is now

$$\Pi \Gamma(\lambda) \Pi = \begin{pmatrix} \Gamma([\lambda] \downarrow S_p) & 0 \\ 0 & 0 \end{pmatrix}$$

where  $\Gamma([\lambda] \downarrow S_p)$  is a  $(p+1) \times (p+1)$  matrix with entries in  $\text{End}(V_\lambda)$ . By (2.5), the expression

$$\sum_{\pi \in NC(n)} \langle \cdot \rangle [\pi](\Pi, \dots, \Pi) R[K(\pi)](\Gamma(\lambda), \dots, \Gamma(\lambda))$$

is the  $n$ th moment of the distribution of  $\tilde{\Pi} X \tilde{\Pi}$ , where  $X$  and  $\tilde{\Pi}$  are free random variables,  $X$  is distributed as  $\Gamma(\lambda)$ , and  $\tilde{\Pi}$  is a projection with normalized trace  $(p+1)/(q+1)$ . In other words, this expression is equal to  $(p+1)/(q+1) \int_{\mathbb{R}} x^n \pi_{(p+1)/(q+1)}(m_\lambda)(dx)$ . It follows from (6.1.1) that

$$\text{tr}_{[\lambda] \downarrow S_p}(\langle \Gamma([\lambda] \downarrow S_p)^n \rangle) = \int_{\mathbb{R}} x^n \pi_{(p+1)/(q+1)}(m_\lambda)(dx) + O(q^{n/2-1})$$

Furthermore, let  $d_\lambda^v$  be the multiplicity of  $[v] \in \mathcal{Y}_p$  in  $[\lambda] \downarrow S_p$ , then one has

$$\begin{aligned} & \text{tr}_{[\lambda] \downarrow S_p} \left( \left( \langle \Gamma([\lambda] \downarrow S_p)^n \rangle - \int_{\mathbb{R}} x^n \pi_{(p+1)/(q+1)}(\mathfrak{m}_\lambda)(dx) \right)^2 \right) \\ &= \sum_{\nu \in \mathcal{A}_p} d_\lambda^\nu \frac{\dim([\nu])}{\dim([\lambda])} \left( \int_{\mathbb{R}} x^n \mathfrak{m}_\nu(dx) - \int_{\mathbb{R}} x^n \pi_{(p+1)/(q+1)}(\mathfrak{m}_\lambda)(dx) \right)^2 \end{aligned}$$

By (6.1.1) and (6.1.2) this last expression is  $O(q^{n-1})$ , uniformly over  $\alpha q \leq p \leq (1 - \alpha)q$ . Arguing as in Section 5.2, applying Markov’s inequality, we see that for most irreducible components  $[\nu]$  of  $[\lambda] \downarrow S_p$ , the moments of the measure  $\mathfrak{m}_\nu$  are close to those of  $\pi_{(p+1)/(q+1)}(\mathfrak{m}_\lambda)$ . For the same reason, for most irreducible components  $[\nu]$  of  $[\lambda] \downarrow S_{q-p}$ , the moments of the measure  $\mathfrak{m}_\nu$  are close to those of  $\pi_{(q-p+1)/(q+1)}(\mathfrak{m}_\lambda)$ , and taking the intersection of these two sets, we see that for most irreducible components  $[\nu_1] \# [\nu_2]$  of  $[\lambda] \downarrow S_p \times S_{q-p}$ , the moments of the measure  $\mathfrak{m}_{\nu_1}$  are close to those of  $\pi_{(p+1)/(q+1)}(\mathfrak{m}_\lambda)$  and the moments of the measure  $\mathfrak{m}_{\nu_2}$  are close to those of  $\pi_{(q-p+1)/(q+1)}(\mathfrak{m}_\lambda)$ . Again we omit the details. ■

The proof of Theorem 1.5.2 is along the same lines as the proof of Theorem 1.4.2.

## 7. REPRESENTATIONS OF SYMMETRIC AND GENERAL LINEAR GROUPS

The last parts of the paper are devoted to the induction problem and the proof of Theorem 3.1.3. It would be interesting to have a proof of this result along the lines of 3.1.1 and 3.1.2, by exhibiting a decomposition of the operator  $\Gamma([\lambda] \circ [\mu])$  into a sum of two operators which would be asymptotically free. One can observe that the obvious decomposition of the Jucys–Murphy element  $[\lambda] \circ [\mu](\mathcal{M}_q)$  into the sum of the first  $|\lambda|$  terms and the last  $|\mu|$  terms does not yield two asymptotically free operators. One can find more subtle decompositions, which are good candidates, but a direct analysis involving only symmetric group has escaped us. In some sense what we do in the next sections is to find another route to this decomposition, using the duality between representations of symmetric groups and general linear groups, and this allows us to exploit large matrix theory.

### 7.1. Schur–Weyl Duality and the Littlewood–Richardson Rule for Outer Products

Besides representations of symmetric groups, a Young diagram  $\lambda_1 \geq \lambda_2 \geq \dots$  can also be used to denote the highest weight of a finite dimensional irreducible representation  $\rho_\lambda$  of  $GL(N, \mathbb{C})$  for  $N$  large enough, occurring in the tensor powers of the basic representation. Indeed, by a

classical result of Schur (see [Wey]), the space  $V^{\otimes q}$  where  $V$  is a complex  $N$ -dimensional space, carrying the basic representation of  $GL(N, \mathbb{C})$ , can be decomposed into a direct sum

$$V^{\otimes q} = \sum_{\lambda \in \mathcal{Y}_q} E_\lambda \quad (7.1.1)$$

where  $\mathcal{Y}_q$  denotes the set of all Young diagrams with  $|\lambda| = q$  and  $E_\lambda$  is a subspace invariant under the natural action of  $S_q \times GL(N, \mathbb{C})$  carrying an irreducible representation of class  $[\lambda] \otimes \rho_\lambda$ .

The Littlewood–Richardson rule gives the multiplicity  $c_{\lambda\mu}^\nu$  of the representation  $[\nu]$  in the decomposition of  $[\lambda] \circ [\mu]$ . These coefficients also arise when taking the Kronecker tensor product of the two representations  $\rho_\lambda$  and  $\rho_\mu$  of  $GL(N, \mathbb{C})$ , namely for  $N$  large enough,  $c_{\lambda\mu}^\nu$  is the multiplicity of  $\rho_\nu$  in the decomposition of  $\rho_\lambda \otimes \rho_\mu$ . This can be seen from Schur–Weyl’s duality relation (7.1.1). We shall not recall the explicit description of the Littlewood–Richardson rule, but rather refer to e.g. [JK], [M] or [R] for several formulations.

## 7.2. Another Spectral Interpretation

We start by quoting the following Lemma which are quite standard. We give the proof for completeness.

**LEMMA 7.2.1.** *The normalized character of the natural representation of  $S_q$  on  $V^{\otimes q}$  where  $V$  is an  $N$ -dimensional space, is  $\chi_N(\sigma) = N^{-|\sigma|}$ .*

*Proof.* Just compute the trace in the basis  $e_{i_1} \otimes \cdots \otimes e_{i_q}$ , where  $e_1, \dots, e_N$  is a basis of  $V$ . ■

**LEMMA 7.2.2.** *Let  $\lambda \in \mathcal{Y}_q$ , then one has*

$$\lim_{N \rightarrow \infty} N^{-q} \dim(\rho_\lambda) = \dim([\lambda])/q!$$

*Proof.* This follows from the explicit formulas for the dimensions of the representations involved (Weyl’s dimension formula for  $\rho_\lambda$  and the hook formula for  $[\lambda]$ ). We give however a more instructive, less computational proof which relies on Lemma 7.2.1. Consider the representation of  $S_q$  on  $V^{\otimes q}$  where  $V$  is an  $N$ -dimensional space carrying the basic representation of  $GL(N, \mathbb{C})$ , then by Lemma 7.2.1, the normalized character of this representation converges, as  $N \rightarrow \infty$ , toward the normalized character of the regular representation of  $S_q$ . It follows that the coefficient of the character of  $[\lambda]$  in  $\chi_N$ , which is equal to  $N^{-q} \dim(\rho_\lambda)$  by (7.1.1), converges as  $N \rightarrow \infty$ , towards the coefficient of  $\chi_\lambda$  in the normalized character of the regular representation, which is  $\dim([\lambda])/q!$ . ■

Let  $E_{ij}; 1 \leq i, j \leq N$  be the canonical basis of the Lie algebra  $\mathfrak{gl}(N, \mathbb{C})$ . Let now  $\lambda$  be a Young diagram and  $N > |\lambda|$ . Denote by  $W_\lambda$  the space of an irreducible representation  $\rho_\lambda$  of the group  $GL(N, \mathbb{C})$  with highest weight  $\lambda$ . We denote by  $d\rho_\lambda$  the derived representation of the Lie algebra  $\mathfrak{gl}(N, \mathbb{C})$ . There is a Hilbert space structure on  $W_\lambda$  such that  $d\rho_\lambda(E_{ij})^* = d\rho_\lambda(E_{ji})$ . We denote by  $\text{tr}_\lambda$  the normalized trace on  $\text{End}(W_\lambda)$  and by  $\langle \cdot \rangle$  the normalized trace on  $M_N(\mathbb{C})$  (this should not cause confusion with the notations we used earlier). Then the von Neumann algebra  $\text{End}(W_\lambda) \otimes M_N(\mathbb{C})$ , endowed with the trace  $\text{tr}_\lambda \otimes \langle \cdot \rangle$ , is a finite dimensional non-commutative probability space. We define a self-adjoint element of  $\text{End}(W_\lambda) \otimes M_N(\mathbb{C})$  by the formula

$$X(\lambda, N) = \sum_{1 \leq i, j \leq N} d\rho_\lambda(E_{ji}) \otimes E_{ij}$$

We shall now give another spectral interpretation of the measure  $m_\lambda$  associated with the Young diagram  $\lambda$ . By a slight abuse of notation, we denote by  $\text{tr}_\lambda$  the trace dilated to  $\text{End}(W_\lambda) \otimes M_N(\mathbb{C})$  by  $\text{tr}_\lambda \otimes Id$ , and similarly for  $\langle \cdot \rangle$ .

**PROPOSITION 7.2.** *For all integer  $k \geq 1$ , one has*

$$\lim_{N \rightarrow \infty} \text{tr}_\lambda(\langle X(\lambda, N)^k \rangle) = \int_{\mathbb{R}} x^k m_\lambda(dx)$$

This means that the measure  $m_\lambda$  is the weak limit, as  $N \rightarrow \infty$ , of the distributions of the elements  $X(\lambda, N)$  in the non-commutative probability spaces  $(\text{End}(W_\lambda) \otimes M_N(\mathbb{C}), \text{tr}_\lambda \otimes \langle \cdot \rangle)$ .

*Proof.* First note that  $\sum_{1 \leq i, j \leq N} E_{ji} \otimes E_{ij}$  acting on  $\mathbb{C}^N \otimes \mathbb{C}^N$ , is the transposition sending  $x \otimes y$  to  $y \otimes x$ . We consider the operator

$$\sum_{k=1}^q \sum_{1 \leq i, j \leq N} (I^{\otimes (k-1)} \otimes E_{ji} \otimes I^{\otimes (q-k)}) \otimes E_{ij}$$

on  $(\mathbb{C}^N)^{\otimes (q+1)}$ . By the above remark, this is the image of the Jucys–Murphy element  $\mathcal{M}_{q+1}$  in the canonical representation of  $S_{q+1}$  on  $(\mathbb{C}^N)^{\otimes (q+1)}$ . Multiplying on the left by the Young projector  $\pi_\lambda$ , one gets the operator  $I \otimes X(\lambda, N)$ , acting on the space  $E_\lambda$  of formula (7.1.1), considered as the  $S_q \times GL(N, \mathbb{C})$  module  $[A] \otimes \rho_\lambda$ . By Lemma 7.2.1, as  $N \rightarrow \infty$ , the character of the representation of  $S_q$  on  $V^{\otimes q}$  converges towards that of the regular representation of  $S_q$ , thus the distribution of  $\pi_\lambda(\mathcal{M}_{q+1})$  on  $E_\lambda$  converges towards  $m_\lambda$  as in the computations of Proposition 3.3. ■

### 7.3. Two Operators

We now elaborate on the last subsection and consider simultaneously two representations associated with Young diagrams  $\lambda$  and  $\mu$ , acting on the spaces  $W_\lambda$  and  $W_\mu$ . Let this time

$$X \equiv X(\lambda, N) = \sum_{1 \leq i, j \leq N} d\rho_\lambda(E_{ji}) \otimes I \otimes E_{ij}$$

and

$$Y \equiv Y(\mu, N) = \sum_{1 \leq i, j \leq N} I \otimes d\rho_\mu(E_{ji}) \otimes E_{ij}$$

be elements of  $End(W_\lambda) \otimes End(W_\mu) \otimes M_N(\mathbb{C})$ .

**PROPOSITION 7.3.** *For all integers  $k_1, \dots, k_r$  one has*

$$\begin{aligned} & \lim_{N \rightarrow \infty} \text{tr}_\lambda \otimes \text{tr}_\mu [\langle (X+Y)^{k_1} \rangle \dots \langle (X+Y)^{k_r} \rangle] \\ &= \sum_{\nu \in \mathcal{A}_{|\lambda|+|\mu|}} c_{\lambda\mu}^\nu \frac{\dim([\nu])}{\dim([\lambda] \circ [\mu])} \left( \int_{\mathbb{R}} x^{k_1} m_\nu(dx) \right) \dots \left( \int_{\mathbb{R}} x^{k_r} m_\nu(dx) \right) \end{aligned}$$

*Proof.* Consider the decomposition of the space  $W_\lambda \otimes W_\mu$  under the action of the representation  $\rho_\lambda \otimes \rho_\mu$  of  $GL(N, \mathbb{C})$ . Each component of type  $\rho_\nu$  occurs with multiplicity  $c_{\lambda\mu}^\nu$ . By Schur's Lemma, on each such component the operators  $\langle (X+Y)^k \rangle$  are diagonal, with eigenvalue  $\text{tr}_\nu(\langle X(\nu, N)^k \rangle)$ . Thus for fixed  $N$  we have

$$\begin{aligned} & \text{tr}_\lambda \otimes \text{tr}_\mu [\langle (X+Y)^{k_1} \rangle \dots \langle (X+Y)^{k_r} \rangle] \\ &= \sum_{\nu \in \mathcal{A}_{|\lambda|+|\mu|}} c_{\lambda\mu}^\nu \frac{\dim(\rho_\nu)}{\dim(\rho_\lambda) \dim(\rho_\mu)} \text{tr}_\nu(\langle X(\nu, N)^{k_1} \rangle) \dots \text{tr}_\nu(\langle X(\nu, N)^{k_r} \rangle) \end{aligned}$$

Applying Lemma 7.2.1 and Proposition 7.2, we get the result, since  $\dim([\lambda] \circ [\mu]) = ((m+n)!)/(n! m!) \dim([\lambda]) \dim([\mu])$ . ■

## 8. CASIMIR ELEMENTS ASSOCIATED WITH PERMUTATIONS

In this section we gather information on permutations and we introduce certain elements in the center of the enveloping algebra of  $\mathfrak{gl}(N, \mathbb{C})$ .

### 8.1. A Fundamental Lemma

We denote by  $Z$  the cycle  $(123 \dots q)$ . The following simple Lemma will be crucial in the proof of asymptotic freeness.

LEMMA 8.1. *Let  $\sigma$  and  $\tau$  be permutations in  $S_q$ , such that  $|\sigma| + |Z\tau^{-1}| + |\tau\sigma^{-1}| = |Z|$ , then either  $\sigma$  or  $Z\tau^{-1}$  has a fixed point.*

*Proof.* Since  $|Z| = q - 1$ , one must have either  $|\sigma| \leq (q - 1)/2$  or  $|Z\tau^{-1}| \leq (q - 1)/2$ . But the product of at most  $(q - 1)/2$  transpositions can move at most  $q - 1$  elements, hence either  $\sigma$  or  $Z\tau^{-1}$  has a fixed point. ■

### 8.2. Anti-exceedances

For  $\sigma \in S_q$  we define  $aexc(\sigma)$  as the number of  $i$  such that  $\sigma(i) \leq i$ , i.e. the number of anti-exceedances of the permutation (see, e.g. [FS]).

LEMMA 8.2. *Let  $1 \leq k \leq q - 1$ , and denote by  $\varepsilon$  the transposition  $(k \ k + 1)$ , then for all  $\sigma, \tau \in S_q$  one has*

$$(1) \quad aexc(\tau) + aexc(Z\tau^{-1}) = q + 1$$

$$(2) \quad |aexc(\sigma) - aexc(\tau)| \leq |\tau\sigma^{-1}|.$$

*Proof.* For every  $j \leq q - 1$  we have the two mutually exclusive possibilities: either  $\tau(j) \leq j$  or  $Z\tau^{-1}(\tau(j)) = j + 1 \leq \tau(j)$ , hence either  $j$  is an anti-exceedance of  $\tau$  or  $\tau(j)$  is an anti-exceedance of  $Z\tau^{-1}$ . Finally one sees that  $q$  is always an anti-exceedance of  $\tau$  and  $\tau(q)$  is an anti-exceedance of  $Z\tau^{-1}$ . Counting all anti-exceedances gives part (1). For the second part, it is easy to check when  $\tau\sigma^{-1}$  is a transposition, and the general case follows by the triangular inequality for the distance  $|\cdot|$ . ■

### 8.3. Definition of the Casimir Elements

Let  $\zeta_{ij}$ ;  $1 \leq i, j \leq N$  be the canonical basis of  $\mathcal{U}(\mathfrak{gl}(N, \mathbb{C}))$ , the universal enveloping algebra of  $\mathfrak{gl}(N, \mathbb{C})$ . Thus the  $\zeta_{ij}$  satisfy the commutation relations

$$[\zeta_{ij}, \zeta_{kl}] = \zeta_{il}\delta_{jk} - \zeta_{kj}\delta_{li}$$

Let  $q$  be a positive integer and for  $\sigma \in S_q$ , define the following element of  $\mathcal{U}(\mathfrak{gl}(N, \mathbb{C}))$

$$C_\sigma = \sum_{1 \leq i_1, \dots, i_q \leq N} \zeta_{i_1 i_{\sigma(1)}} \zeta_{i_2 i_{\sigma(2)}} \cdots \zeta_{i_q i_{\sigma(q)}}$$

In order to ease notations, we shall not indicate the dependence in  $N$  of these elements in the sequel. When  $\sigma$  is the cycle  $(123 \cdots q)$  we put  $C_\sigma = C_q$ .

One can easily check that  $C_\sigma$  belongs to the center of  $\mathcal{U}(\mathfrak{gl}(N, \mathbb{C}))$ . Let us now introduce a convenient notation. For an integer  $p \geq 1$  define

$$\zeta_{ij}^{(p)} = \sum_{1 \leq i_1, \dots, i_{p-1} \leq N} \zeta_{i i_1} \zeta_{i_1 i_2} \cdots \zeta_{i_{p-2} i_{p-1}} \zeta_{i_{p-1} j} \in \mathcal{U}(\mathfrak{gl}(N, \mathbb{C}))$$

These elements satisfy the commutation relations

$$[\zeta_{ij}^{(p)}, \zeta_{kl}^{(q)}] = \zeta_{il}^{(p+q-1)} \delta_{jk} - \zeta_{kj}^{(p+q-1)} \delta_{li}.$$

Let  $\mathbf{p} = (p_1, \dots, p_q)$  where  $p_1, \dots, p_q$  are positive integers, and  $|\mathbf{p}| = p_1 + \dots + p_q$ . If  $\tau$  is a permutation of  $\{1, \dots, q\}$  let  $\mathbf{p}^\tau$  be the vector  $(p_{\tau(1)}, \dots, p_{\tau(q)})$ . We now define the following Casimir operator

$$C_\sigma^{(\mathbf{p})} = \sum_{1 \leq i_1, \dots, i_q \leq N} \zeta_{i_1 i_{\sigma(1)}}^{(p_1)} \zeta_{i_2 i_{\sigma(2)}}^{(p_2)} \dots \zeta_{i_q i_{\sigma(q)}}^{(p_q)}$$

Replacing the elements  $\zeta_{ij}^{(p_k)}$  by their defining expression, we see that in fact  $C_\sigma^{(\mathbf{p})} = C_{\sigma'}$  where  $\sigma'$  is a permutation of  $|\mathbf{p}|$  elements, obtained from  $\sigma$  by inserting  $p_{k-1}$  elements in the cycle between  $k$  and  $\sigma(k)$ .

LEMMA 8.3. *Let  $\sigma \in S_q$  and for some  $1 \leq r \leq q-1$ , let  $\varepsilon$  be the transposition  $(r \ r+1)$ .*

(1) *If  $\sigma(r) = r+1$  then*

$$C_\sigma^{(\mathbf{p})} = C_{\sigma'}^{(\mathbf{p}')$$

where  $\mathbf{p}' = (p_1, \dots, p_{r-1}, p_r + p_{r+1}, p_{r+2}, \dots, p_q)$  and  $\sigma'$  is the permutation of  $\{1, \dots, q-1\}$  given by  $\sigma' = \alpha\sigma\beta$ , where  $\alpha(j) = j$ ;  $\beta(j) = j$  for  $j \leq r$ ,  $\alpha(j) = j-1$ ;  $\beta(j) = j+1$  for  $j > r$ .

(2) *If  $\sigma(r) \neq r+1$  and  $\sigma(r+1) = r$ , then*

$$C_\sigma^{(\mathbf{p})} - C_{\varepsilon\sigma\varepsilon^{-1}}^{(\mathbf{p}^\varepsilon)} = C_{\sigma'}^{(\mathbf{p}')} - NC_{\sigma''}^{(\mathbf{p}'')}$$

where  $\mathbf{p}' = \mathbf{p}'' = (p_1, \dots, p_{r-1}, p_r + p_{r+1} - 1, p_{r+2}, \dots, p_q)$ ,  $\sigma'$  is the permutation of  $\{1, \dots, q-1\}$  given by  $\sigma' = \alpha'\sigma\beta'$ , where  $\alpha'(j) = j$ ;  $\beta'(j) = j$  for  $j < r$ ,  $\alpha'(r) = r$ ;  $\beta'(r) = r+1$ ,  $\alpha'(j) = j-1$ ;  $\beta'(j) = j+1$  for  $j > r$ , and  $\sigma''$  is the permutation of  $\{1, \dots, q-1\}$  given by  $\sigma'' = \alpha''\sigma\beta''$ , where  $\alpha''(j) = j$ ;  $\beta''(j) = j$  for  $j \leq r$ ,  $\alpha''(j) = j-1$ ;  $\beta''(j) = j+1$  for  $j > r$

(3) *If  $\sigma(r) \neq r+1$  and  $\sigma(r+1) \neq r$ , then*

$$C_\sigma^{(\mathbf{p})} - C_{\varepsilon\sigma\varepsilon^{-1}}^{(\mathbf{p}^\varepsilon)} = C_{\sigma'}^{(\mathbf{p}')} - C_{\sigma''}^{(\mathbf{p}'')}$$

where  $\mathbf{p}'$ ,  $\mathbf{p}''$ ,  $\sigma'$  and  $\sigma''$  are as in (2).

*Proof.* The first case of the lemma is easy. For the other cases, observe that for any  $\sigma \in S_q$  one has

$$C_{\varepsilon\sigma\varepsilon^{-1}}^{(\mathbf{p}^\varepsilon)} = \sum_{1 \leq i_1, \dots, i_q \leq N} \zeta_{i_1 i_{\varepsilon\sigma(1)}}^{(p_1)} \dots \zeta_{i_{r-1} i_{\varepsilon\sigma(r-1)}}^{(p_{r-1})} \zeta_{i_r i_{\varepsilon\sigma(r+1)}}^{(p_{r+1})} \zeta_{i_{r+1} i_{\varepsilon\sigma(r)}}^{(p_r)} \zeta_{i_{r+2} i_{\varepsilon\sigma(r+2)}}^{(p_{r+2})} \dots \zeta_{i_q i_{\varepsilon\sigma(q)}}^{(p_q)}$$

Exchanging the numbering of the indices  $i_r, i_{r+1}$  in the above sum gives

$$C_{\varepsilon\sigma\varepsilon^{-1}}^{(\mathbf{p}^\varepsilon)} = \sum_{1 \leq i_1, \dots, i_q \leq N} \zeta_{i_1 i_{\sigma(1)}}^{(p_1)} \cdots \zeta_{i_{r-1} i_{\sigma(r-1)}}^{(p_{r-1})} \zeta_{i_{r+1} i_{\sigma(r+1)}}^{(p_{r+1})} \zeta_{i_r i_{\sigma(r)}}^{(p_r)} \zeta_{i_{r+2} i_{\sigma(r+2)}}^{(p_{r+2})} \cdots \zeta_{i_q i_{\sigma(q)}}^{(p_q)}$$

an application of the commutation relations shows that

$$\begin{aligned} C_\sigma^{(\mathbf{p})} - C_{\varepsilon\sigma\varepsilon^{-1}}^{(\mathbf{p}^\varepsilon)} &= \sum_{1 \leq i_1, \dots, i_q \leq N} \zeta_{i_1 i_{\sigma(1)}}^{(p_1)} \cdots [\zeta_{i_r i_{\sigma(r)}}^{(p_r)}, \zeta_{i_{r+1} i_{\sigma(r+1)}}^{(p_{r+1})}] \cdots \zeta_{i_q i_{\sigma(q)}}^{(p_q)} \\ &= \sum_{1 \leq i_1, \dots, i_q \leq N} \zeta_{i_1 i_{\sigma(1)}}^{(p_1)} \cdots \zeta_{i_{r-1} i_{\sigma(r-1)}}^{(p_{r-1})} \zeta_{i_r i_{\sigma(r+1)}}^{(p_r + p_{r+1} - 1)} \\ &\quad \times \delta_{i_{r+1} i_{\sigma(r)}} \zeta_{i_{r+2} i_{\sigma(r+2)}}^{(p_{r+2})} \cdots \zeta_{i_q i_{\sigma(q)}}^{(p_q)} \\ &\quad - \sum_{1 \leq i_1, \dots, i_q \leq N} \zeta_{i_1 i_{\sigma(1)}}^{(p_1)} \cdots \zeta_{i_{r-1} i_{\sigma(r-1)}}^{(p_{r-1})} \zeta_{i_{r+1} i_{\sigma(r)}}^{(p_r + p_{r+1} - 1)} \\ &\quad \times \delta_{i_r i_{\sigma(r+1)}} \zeta_{i_{r+2} i_{\sigma(r+2)}}^{(p_{r+2})} \cdots \zeta_{i_q i_{\sigma(q)}}^{(p_q)} \end{aligned}$$

Parts (2) and (3) of the lemma then follow by inspection. ■

#### 8.4. Expression in Terms of Basic Casimir Elements

Since the elements  $C_\sigma$  are in the center of  $\mathcal{U}(\mathfrak{gl}(N, \mathbb{C}))$ , they can be expressed in terms of the basic Casimir operators  $C_q$ , indeed one has

LEMMA 8.4.1. *For all  $\sigma \in S_q$  there exists a unique polynomial  $\mathbf{P}_\sigma$  in  $q + 1$  variables, with integer coefficients, such that for all  $N > 0$ ,*

$$C_\sigma = \mathbf{P}_\sigma(N, C_1, \dots, C_q)$$

*Proof.* We prove the existence by induction on  $q$ . Let  $1 \leq r \leq q - 1$ , and  $\varepsilon$  be the transposition  $(r \ r + 1)$ , then by Lemma 8.3 and the induction hypothesis,  $C_\sigma - C_{\varepsilon\sigma\varepsilon^{-1}}$  is a polynomial in  $N, C_1, C_2, \dots, C_{q-1}$ . Iterating this we see that for any permutation  $\tau$ , conjugate with  $\sigma$ , the same is true of  $C_\sigma - C_\tau$ . Now if  $k_1, k_2, \dots, k_r$  are the lengths of the cycles of  $\sigma$ , then we can easily compute  $C_\tau$  for  $\tau$  the permutation with cycles

$$\begin{aligned} &[12 \cdots k_1]; [(k_1 + 1)(k_1 + 2) \cdots (k_1 + k_2)]; \dots; \\ &[(k_1 + k_2 + \cdots k_{r-1} + 1)(k_1 + k_2 + \cdots k_{r-1} + 2) \cdots \\ &\quad (k_1 + k_2 + \cdots k_{r-1} + 1_r)] \end{aligned}$$

which is conjugate to  $\sigma$ , and get  $C_\tau = C_{k_1} C_{k_2} \cdots C_{k_r}$ . This proves the existence of the polynomial  $\mathbf{P}_\sigma$ . The uniqueness follows from the fact that for all  $N \geq q$  the  $C_1, \dots, C_q$  are algebraically free in  $\mathcal{U}(\mathfrak{gl}(N, \mathbb{C}))$ . ■

COROLLARY 8.4.2. *There exist polynomials  $\mathbf{P}_\sigma^{(\mathbf{p})}$  such that, for all  $N > 0$ ,*

$$C_\sigma^{(\mathbf{p})} = \mathbf{P}_\sigma^{(\mathbf{p})}(N, C_1, \dots, C_{|\mathbf{p}|})$$

### 8.5. Gradations

We now define three gradations on the space of polynomials in the variables  $N, C_1, \dots, C_q, \dots$ . The first is the degree  $d(\mathbf{P})$  obtained by assigning a degree 1 to each variable  $N, C_1, \dots, C_q, \dots$ . The second is  $h(\mathbf{P})$ , obtained by assigning a degree zero to  $N$  and a degree  $j$  to  $C_j$ . The third is  $v(\mathbf{P})$ , obtained by assigning degree 1 to  $N$  and degree  $j$  to  $C_j$ .

Let  $\sigma$  be a permutation, for each cycle  $c$  of the permutation we denote by  $p_c$  the sum  $\sum_{j \in c} p_j$ . We define polynomials in several variables  $\mathbf{R}_\sigma^{(\mathbf{p})}(x_1, \dots, x_{|\mathbf{p}|})$  by the formula

$$\mathbf{R}_\sigma^{(\mathbf{p})}(x_1, \dots, x_{|\mathbf{p}|}) = \prod_{c|\sigma} x_{p_c}$$

where the product is over the cycles of  $\sigma$ .

We now come to the main technical result of this section.

PROPOSITION 8.5. *For all permutation  $\sigma$  of  $\{1, \dots, q\}$  and vector  $\mathbf{p} = (p_1, \dots, p_q)$  one has*

- (1)  $h(\mathbf{P}_\sigma^{(\mathbf{p})}) = |\mathbf{p}|$
- (2)  $v(\mathbf{P}_\sigma^{(\mathbf{p})}) = |\mathbf{p}|$
- (3)  $d(\mathbf{P}_\sigma^{(\mathbf{p})}) = aexc(\sigma)$

Furthermore,  $\mathbf{P}_\sigma^{(\mathbf{p})}(N, C_1, \dots, C_{|\mathbf{p}|}) - \mathbf{R}_\sigma^{(\mathbf{p})}(C_1, \dots, C_{|\mathbf{p}|})$  has  $h$ -degree  $\leq |\mathbf{p}| - 1$ .

*Proof.* We first prove (1) and (2) by induction on  $q$ . Let  $\varepsilon$  be a transposition ( $r \ r + 1$ ). Using Lemma 8.3, we see that  $\mathbf{P}_\sigma^{(\mathbf{p})} - \mathbf{P}_{\varepsilon\sigma\varepsilon}^{(\mathbf{p})}$  is of the form  $A_N \mathbf{P}_{\sigma'}^{(\mathbf{p})} - B_N \mathbf{P}_{\sigma''}^{(\mathbf{p})}$  where  $\sigma'$  and  $\sigma''$  are permutations of  $q - 1$  elements, one has  $|\mathbf{p}'| = |\mathbf{p}''| = |\mathbf{p}| - 1$ , and  $A_N, B_N$  are either 1 or  $N$ . It follows, by induction hypothesis, that the  $h$ -degree of  $\mathbf{P}_\sigma^{(\mathbf{p})} - \mathbf{P}_{\varepsilon\sigma\varepsilon}^{(\mathbf{p})}$  is  $\leq |\mathbf{p}| - 1$  and its  $v$ -degree is  $\leq |\mathbf{p}|$ . Replacing  $\varepsilon$  by any permutation, this holds again, hence if we look at  $\tau = \rho\sigma\rho^{-1}$  the permutation introduced in the proof of Lemma 8.4.1, we see that  $\mathbf{P}_\sigma^{(\mathbf{p})} - \mathbf{P}_{\rho\sigma\rho^{-1}}^{(\mathbf{p})}$  has  $h$ -degree  $\leq |\mathbf{p}| - 1$  and  $v$ -degree  $\leq |\mathbf{p}|$ . But  $\mathbf{P}_{\rho\sigma\rho^{-1}}^{(\mathbf{p})}(N, C_1, \dots, C_{|\mathbf{p}|}) = \mathbf{R}_\sigma^{(\mathbf{p})}(C_1, \dots, C_{|\mathbf{p}|})$ , as one can check, so that the  $h$ -degree and the  $v$ -degree of  $\mathbf{P}_{\rho\sigma\rho^{-1}}^{(\mathbf{p})}$  are both  $|\mathbf{p}|$ , hence, the  $h$ -degree of  $\mathbf{P}_\sigma^{(\mathbf{p})}$  is  $|\mathbf{p}|$ , and its  $v$ -degree is  $\leq |\mathbf{p}|$ . Since the  $v$  degree is always larger than the  $h$ -degree, we have the conclusion. The last assertion also follows from these considerations.

We now turn to the proof of part (3) of the Lemma. We shall prove by induction on  $q$  and  $q - \sigma(q)$  that  $\mathbf{P}_\sigma^{(\mathbf{p})}$  has  $d$ -degree equal to  $aexc(\sigma)$ , and more precisely, that it contains a monomial with  $d$ -degree  $aexc(\sigma)$  and with

$v$ -degree  $|\mathbf{p}|$ . Assume that this is true for all permutations of less than  $q - 1$  element.

Let  $\sigma \in S_q$  be such that  $\sigma(q) = q$ , then  $C_\sigma^{(\mathbf{p})} = C_{p_q} C_{\sigma'}^{(\mathbf{p}' )}$  where  $\sigma'$  is the restriction of  $\sigma$  to  $\{1, \dots, q - 1\}$ , and  $\mathbf{p}' = (p_1, \dots, p_{q-1})$ . By induction hypothesis we see that  $\mathbf{P}_\sigma$  has a term of  $d$ -degree  $aexc(\sigma) = aexc(\sigma') + 1$  and of  $v$ -degree  $|\mathbf{p}|$ .

Consider now the case where  $\sigma(q) = q - 1$  and  $\sigma(q - 1) = q$ , then one has  $C_\sigma^{(\mathbf{p})} = C_{p_q + p_{q-1}} C_{\sigma''}^{(\mathbf{p}'')}$  where  $\sigma''$  is the restriction of  $\sigma$  to  $\{1, \dots, q - 2\}$ , and where  $\mathbf{p}'' = (p_1, \dots, p_{q-2})$ . Using the induction hypothesis again we see that  $\mathbf{P}_\sigma$  has a term of  $d$ -degree  $aexc(\sigma) = aexc(\sigma'') + 1$  and of  $v$ -degree  $|\mathbf{p}|$ .

Assume now that  $\sigma(q) = q - 1$  and  $\sigma(q - 1) < q - 1$ , and let  $\varepsilon$  be the transposition  $(q \ q - 1)$ , then using (3) of Lemma 8.3 we see that

$$C_\sigma^{(\mathbf{p})} - C_{\varepsilon\sigma\varepsilon^{-1}}^{(\mathbf{p}^\varepsilon)} = C_{\sigma'}^{(\mathbf{p}' )} - NC_{\sigma''}^{(\mathbf{p}'')}$$

Since  $\varepsilon\sigma\varepsilon^{-1}(q - 1) = q$ , and  $aexc(\varepsilon\sigma\varepsilon^{-1}) = aexc(\sigma) - 1$ , by induction hypothesis, and (1) of Lemma 8.3,  $C_{\varepsilon\sigma\varepsilon^{-1}}^{(\mathbf{p}^\varepsilon)}$  has  $d$ -degree equal to  $aexc(\sigma) - 1$ . One sees that  $C_{\sigma'}^{(\mathbf{p}' )}$  and  $C_{\sigma''}^{(\mathbf{p}'')}$  have  $v$ -degree  $|\mathbf{p}| - 1$ . By induction hypothesis,  $C_{\sigma''}^{(\mathbf{p}'')}$  contains a term with  $d$ -degree  $aexc(\sigma'') = aexc(\sigma) - 1$  and  $v$ -degree  $|\mathbf{p}| - 1$ , hence  $NC_{\sigma''}^{(\mathbf{p}'')}$  contains a term with  $d$ -degree  $aexc(\sigma)$  and  $v$ -degree  $|\mathbf{p}|$ , and the induction hypothesis holds for  $\sigma$ .

Finally we assume that  $\sigma(q) = r < q - 1$  and let  $\varepsilon$  be the transposition  $(r \ r + 1)$ , then  $\sigma(r + 1) \neq r$ . If  $\sigma(r) = r + 1$  then we can apply (1) of Lemma 8.3, and we obtain easily the result. If  $\sigma(r) \neq r + 1$ , we apply part 3 of Lemma 8.3 and we see that

$$C_\sigma^{(\mathbf{p})} - C_{\varepsilon\sigma\varepsilon^{-1}}^{(\mathbf{p}^\varepsilon)} = C_{\sigma'}^{(\mathbf{p}' )} - C_{\sigma''}^{(\mathbf{p}'')}$$

This time the two terms on the right have  $v$ -degree  $|\mathbf{p}| - 1$ , moreover, we have  $q - \sigma(q) = q - \varepsilon\sigma\varepsilon^{-1}(q) + 1$ , and  $aexc(\varepsilon\sigma\varepsilon^{-1}) = aexc(\sigma)$ . Using induction on  $q - \sigma(q)$ , we know that  $C_{\varepsilon\sigma\varepsilon^{-1}}^{(\mathbf{p}^\varepsilon)}$  contains a term with  $d$ -degree  $aexc(\sigma)$  and  $v$ -degree  $|\mathbf{p}|$ , and the claim follows. ■

### 8.6. Limit Polynomials

Once we have computed the  $d$ -degree of  $\mathbf{P}_\sigma^{(\mathbf{p})}$ , we can define new polynomials in several variables by the formula

$$\mathbf{Q}_\sigma^{(\mathbf{p})}(x_1, \dots, x_{|\mathbf{p}|}) = \lim_{N \rightarrow \infty} N^{-d(\mathbf{P}_\sigma^{(\mathbf{p})})} \mathbf{P}_\sigma^{(\mathbf{p})}(N, Nx_1, \dots, Nx_{|\mathbf{p}|})$$

Note that only terms of highest  $d$ -degree in  $\mathbf{P}_\sigma^{(\mathbf{p})}$  give rise to terms in the limit polynomial  $\mathbf{Q}_\sigma^{(\mathbf{p})}$ . We define the  $h$ -degree of such a polynomial by assigning degree  $j$  to the variable  $x_j$ .

**PROPOSITION 8.6.** *For any permutation  $\sigma \in S_q$ , one has  $aexc(\sigma) + |\sigma| \geq q$ .*

*If  $aexc(\sigma) + |\sigma| > q$ , then  $h(\mathbf{Q}_\sigma^{(\mathbf{p})}) \leq |\mathbf{p}| - 1$*

*If  $aexc(\sigma) + |\sigma| = q$ , then  $h(\mathbf{Q}_\sigma^{(\mathbf{p})} - \mathbf{R}_\sigma^{(\mathbf{p})}) \leq |\mathbf{p}| - 1$*

*Proof.* First we note that each cycle of  $\sigma$  contains a  $j$  such that  $\sigma(j) \leq j$ , so that  $aexc(\sigma) \geq c(\sigma)$ , this gives the first assertion. For the other assertions, we have to compute the terms in  $\mathbf{P}_\sigma^{(\mathbf{p})}$  of  $h$ -degree  $|\mathbf{p}|$  and  $d$ -degree  $aexc(\sigma)$ . By the last assertion in Lemma 8.5, the only term of  $h$ -degree  $|\mathbf{p}|$  in  $\mathbf{P}_\sigma^{(\mathbf{p})}(N, C_1, \dots, C_{|\mathbf{p}|})$  is  $\mathbf{R}_\sigma^{(\mathbf{p})}(C_1, \dots, C_{|\mathbf{p}|})$ . The  $d$ -degree of this term is equal to  $c(\sigma)$ , hence it gives a contribution to  $\mathbf{Q}_\sigma^{(\mathbf{p})}$  only if  $c(\sigma) = aexc(\sigma)$ . ■

## 9. EXPANSION OF UNITARY GROUP INTEGRALS

### 9.1. Asymptotics of Integrals

First we recall a result on asymptotic evaluation of unitary group integrals. Such results have been obtained for classical groups by Weingarten, see [Wei]. The full proof in the unitary group case can be found in [X]. We denote by  $dU$  the normalized Haar measure on  $U(N)$ , and by  $U_{ij}$  the coordinate functions.

**PROPOSITION 9.1.** *Let  $i_1, \dots, i_q, j_1, \dots, j_q, i'_1, \dots, i'_q, j'_1, \dots, j'_q$ , be positive integers, then one has*

$$\begin{aligned} & \int_{U(N)} U_{i_1 j_1} \cdots U_{i_q j_q} \bar{U}_{j'_1 i'_1} \cdots \bar{U}_{j'_q i'_q} dU \\ &= \sum_{\sigma, \tau \in S_q} N^{-q - |\tau\sigma^{-1}|} \delta_{i_1 i'_{\sigma(1)}} \cdots \delta_{i_q i'_{\sigma(q)}} \delta_{j_1 j'_{\tau(1)}} \cdots \delta_{j_q j'_{\tau(q)}} \phi(N, q, \tau\sigma^{-1}) \end{aligned}$$

where  $\phi(N, q, \tau\sigma^{-1})$  is independent of the indices  $i_1, \dots, j'_q$ , and converges towards  $\phi(q, \tau\sigma^{-1}) \in \mathbb{R}$  as  $N \rightarrow \infty$ . Moreover,  $\phi(q, \tau\sigma^{-1})$  depends only on the conjugacy class of  $\tau\sigma^{-1}$  in  $S_q$ .

*Proof.* See Lemma 2.2 of [X]. ■

Beware that here  $|\tau\sigma^{-1}| = q - c(\tau\sigma^{-1})$  does not have the same significance as in [X].

### 9.2. Unitary Invariance of Representations

Let  $\lambda$  and  $\mu$  be Young diagrams, we consider the operators  $X(\lambda, N)$  and  $Y(\mu, N)$  introduced in Section 7.3. Let  $1 \leq q_1 < q_2 < \cdots < q_r$  and

$\mathbf{p} = (p_1, \dots, p_{q_r})$ ,  $\mathbf{s} = (s_1, \dots, s_{q_r})$ , be positive integers, we shall compute, with the help of Proposition 9.1, the limit as  $N \rightarrow \infty$  of

$$\begin{aligned} & \text{tr}_\lambda \otimes \text{tr}_\mu (\langle X^{p_1} Y^{s_1} X^{p_2} \dots X^{p_{q_1}} Y^{s_{q_1}} \rangle \langle X^{p_{q_1+1}} Y^{s_{q_1+1}} X^{p_{q_1+2}} \dots X^{p_{q_2}} Y^{s_{q_2}} \rangle \dots \\ & \langle X^{p_{q_{r-1}+1}} Y^{s_{q_{r-1}+1}} X^{p_{q_{r-1}+2}} \dots X^{p_{q_r}} Y^{s_{q_r}} \rangle ) \end{aligned}$$

We shall do the computation for  $r = 2$ , but the same idea works for all  $r$ . First we notice that if  $\Gamma$  is invertible in  $\text{End}(V_\lambda \otimes V_\mu)$ , then for all  $A \in M_N(\mathbb{C}) \otimes \text{End}(V_\lambda \otimes V_\mu)$  one has

$$\Gamma \langle A \rangle \Gamma^{-1} = \langle (1 \otimes \Gamma) A (1 \otimes \Gamma^{-1}) \rangle$$

hence

$$\begin{aligned} & \text{tr}_\lambda \otimes \text{tr}_\lambda (\langle A \rangle \langle B \rangle) \\ & = \text{tr}_\lambda \otimes \text{tr}_\mu (\langle (1 \otimes \Gamma) A (1 \otimes \Gamma^{-1}) \rangle \langle (1 \otimes \Gamma) B (1 \otimes \Gamma^{-1}) \rangle) \end{aligned}$$

We apply this to

$$\begin{aligned} A &= X^{p_1} Y^{s_1} \dots X^{p_{q_1}} Y^{s_{q_1}} \\ B &= X^{p_{q_1+1}} Y^{s_{q_1+1}} \dots X^{p_{q_2}} Y^{s_{q_2}} \\ \Gamma &= \rho_\lambda(U) \otimes I_{V_\mu} \end{aligned}$$

where  $U \in U(N)$ . Since  $1 \otimes \Gamma$  commutes with  $Y$ , we have (identifying, by a slight abuse of notations,  $\rho_\lambda(U)$  with its ampliation  $I_{\mathbb{C}^N} \otimes \rho_\lambda(U) \otimes I_{V_\lambda}$ )

$$\begin{aligned} & \text{tr}_\lambda \otimes \text{tr}_\mu (\langle X^{p_1} Y^{s_1} \dots X^{p_{q_1}} Y^{s_{q_1}} \rangle \langle X^{p_{q_1+1}} Y^{s_{q_1+1}} \dots X^{p_{q_2}} Y^{s_{q_2}} \rangle) \\ & = \text{tr}_\lambda \otimes \text{tr}_\mu (\langle \rho_\lambda(U) X^{p_1} \rho_\lambda(U^*) Y^{s_1} \dots \rho_\lambda(U) X^{p_{q_1}} \rho_\lambda(U^*) Y^{s_{q_1}} \rangle \\ & \quad \times \langle \rho_\lambda(U) X^{p_{q_1+1}} \rho_\lambda(U^*) Y^{s_{q_1+1}} \dots \rho_\lambda(U) X^{p_{q_2}} \rho_\lambda(U^*) Y^{s_{q_2}} \rangle ) \end{aligned}$$

We shall integrate both sides with respect to  $dU$  and investigate the limit of these expressions as  $N \rightarrow \infty$ , with the help of Proposition 9.1. For this we shall do computations in  $\mathcal{U}(\mathfrak{gl}(N, \mathbb{C})) \otimes \mathcal{U}(\mathfrak{gl}(N, \mathbb{C}))$ , where we define elements

$$\alpha_{ij}^{(p)} = \zeta_{ij}^{(p)} \otimes I \quad \text{and} \quad \beta_{ij}^{(p)} = I \otimes \zeta_{ij}^{(p)}$$

Observe that

$$\rho_\lambda(U) X^p \rho_\lambda(U^*) = \sum_{1 \leq i, j, k, l \leq N} U_{jk} d\rho_\lambda(\zeta_{kl}^{(p)}) \bar{U}_{il} \otimes E_{ij}$$

The expression

$$\mathrm{tr}_\lambda \otimes \mathrm{tr}_\mu (\langle X^{p_1} Y^{s_1} \dots X^{p_{q_1}} Y^{s_{q_1}} \rangle \langle X^{p_{q_1+1}} Y^{s_{q_1+1}} \dots X^{p_{q_2}} Y^{s_{q_2}} \rangle)$$

is the image by  $(\mathrm{tr}_\lambda \otimes \mathrm{tr}_\mu) \circ (d\rho_\lambda \otimes d\rho_\mu)$  of the element of  $\mathcal{U}(\mathrm{gl}(N, \mathbb{C})) \otimes \mathcal{U}(\mathrm{gl}(N, \mathbb{C}))$

$$\begin{aligned} & N^{-2} \int_{U(N)} \sum_{\substack{1 \leq i_1, \dots, i_{q_2} i'_1, \dots, i'_{q_2} \leq N \\ 1 \leq j_1, \dots, j_{q_2} j'_1, \dots, j'_{q_2} \leq N}} U_{i_1 j_1} \alpha_{j_1 i'_1}^{(p_1)} \bar{U}_{i'_1 j'_1} \beta_{i'_1 i_2}^{(s_1)} U_{i_2 j_2} \alpha_{j_2 i'_2}^{(p_2)} \dots \\ & \bar{U}_{i'_{q_1} j'_{q_1}} \beta_{i'_{q_1} i_1}^{(s_{q_1})} U_{i_{q_1+1} j_{q_1+1}} \alpha_{j_{q_1+1} i'_{q_1+1}}^{(p_{q_1+1})} \bar{U}_{i'_{q_1+1} j'_{q_1+1}} \beta_{i'_{q_1+1} i_{q_1+2}}^{(s_{q_1+1})} U_{i_{q_1+2} j_{q_1+2}} \dots \\ & \bar{U}_{i'_{q_2} j'_{q_2}} \beta_{i'_{q_2} i_{q_1+1}}^{(s_{q_2})} dU \end{aligned}$$

Using Lemma 9.1, and the fact that the elements  $\alpha_{ij}^{(p)}$  commute with the elements  $\beta_{kl}^{(p')}$ , we see that this expression is equal to

$$\begin{aligned} & \sum_{\sigma, \tau \in S_{q_2}} N^{-q_2-2-|\tau\sigma^{-1}|} \phi(N, q_2, |\tau\sigma^{-1}|) \\ & \times \sum_{\substack{1 \leq i_1, \dots, i_{q_2} i'_1, \dots, i'_{q_2} \leq N \\ 1 \leq j_1, \dots, j_{q_2} j'_1, \dots, j'_{q_2} \leq N}} \alpha_{j_1 i'_{\sigma(1)}}^{(p_1)} \dots \alpha_{j_{q_2} i'_{\sigma(q_2)}}^{(p_{q_2})} \beta_{i_{\tau(1)} i_2}^{(s_1)} \dots \beta_{i_{\tau(q_1)} i_1}^{(s_{q_1})} \beta_{i_{\tau(q_1+1)} i_{q_1+2}}^{(s_{q_1+1})} \dots \beta_{i_{\tau(q_2)} i_{q_1+1}}^{(s_{q_2})} \\ & = \sum_{\sigma, \tau \in S_{q_2}} N^{-q_2-2-|\tau\sigma^{-1}|} \phi(N, q_2, \tau\sigma^{-1}) C_\sigma^{(\mathbf{p})} \otimes C_{W\tau^{-1}}^{(\mathbf{s})} \end{aligned}$$

where  $W$  is the permutation with cycles  $([123 \dots q_1]; [q_1+1 q_1+2 \dots q_2])$ . Applying  $(\mathrm{tr}_\lambda \otimes \mathrm{tr}_\mu) \circ (d\rho_\lambda \otimes d\rho_\mu)$  to this expression, we get

$$\begin{aligned} & \sum_{\sigma, \tau \in S_{q_2}} N^{-q_2-2-|\tau\sigma^{-1}|} \phi(N, q_2, \tau\sigma^{-1}) \\ & \times \mathbf{P}_\sigma^{(\mathbf{p})}(N, N \mathrm{tr}_\lambda(\langle X(\lambda, N) \rangle), \dots, N \mathrm{tr}_\lambda(\langle X(\lambda, N)^{|\mathbf{p}|} \rangle)) \\ & \times \mathbf{P}_{W\tau^{-1}}^{(\mathbf{s})}(N, N \mathrm{tr}_\mu(\langle X(\mu, N) \rangle), \dots, N \mathrm{tr}_\mu(\langle X(\mu, N)^{|\mathbf{s}|} \rangle)) \end{aligned}$$

Bearing in mind that  $\mathrm{tr}_\lambda(\langle X(\lambda, N)^j \rangle) \rightarrow \int_{\mathbb{R}} x^j m_\lambda(dx)$  as  $N \rightarrow \infty$ , by Proposition 7.2 and Section 8.5, a typical term in the sum will be of the order

$$\begin{aligned} & N^{-q_2-2-|\tau\sigma^{-1}|+a\mathrm{exc}(\sigma)+a\mathrm{exc}(W\tau^{-1})} \phi(q_2, \tau\sigma^{-1}) \\ & \times \mathbf{Q}_\sigma^{(\mathbf{p})} \left( \int_{\mathbb{R}} x m_\lambda(dx), \dots, \int_{\mathbb{R}} x^{|\mathbf{p}|} m_\lambda(dx) \right) \\ & \times \mathbf{Q}_{W\tau^{-1}}^{(\mathbf{s})} \left( \int_{\mathbb{R}} x m_\mu(dx), \dots, \int_{\mathbb{R}} x^{|\mathbf{s}|} m_\mu(dx) \right) \end{aligned}$$

For short we shall write  $\mathbf{Q}_\sigma^{(\mathbf{p})}(\lambda)$  instead of  $\mathbf{Q}_\sigma^{(\mathbf{p})}(\int_{\mathbb{R}} x m_\lambda(dx), \dots, \int_{\mathbb{R}} x^{|\mathbf{p}|} m_\lambda(dx))$  in the sequel.

Similarly we shall use  $\mathbf{R}_\sigma^{(\mathbf{p})}(\lambda)$  instead of  $\mathbf{R}_\sigma^{(\mathbf{p})}(\int_{\mathbb{R}} x m_\lambda(dx), \dots, \int_{\mathbb{R}} x^{|\mathbf{p}|} m_\lambda(dx))$ .

By Lemma 8.2, one has  $-q_2 - 2 - |\sigma^{-1}\tau| + aexc(\sigma) + aexc(\sigma) + aexc(W\tau^{-1}) \leq 0$  for all permutations  $\sigma, \tau \in S_{q_2}$ . Indeed, one has  $W = Z\epsilon$  where  $Z = (1 \cdots q_2)$  and  $\epsilon$  is the transposition  $(q_1 q_2)$ , and then

$$\begin{aligned} aexc(W\tau^{-1}) &= aexc(Z\epsilon\tau^{-1}) \\ &= q_2 + 1 - aexc(\tau\epsilon) && \text{(by (1) of Lemma 8.2)} \\ &\leq q_2 + 2 - aexc(\tau) && \text{(by (2) of Lemma 8.2)} \\ &\leq q_2 + 2 + |\sigma^{-1}\tau| - aexc(\sigma) && \text{by (1) again} \end{aligned}$$

We can thus take the limit term by term and we have proved

LEMMA 9.2.1. *For all  $\mathbf{p} = (p_1, \dots, p_{q_2})$  and  $\mathbf{s} = (s_1, \dots, s_{q_2})$ , one has*

$$\begin{aligned} &\lim_{N \rightarrow \infty} \text{tr}_\lambda \otimes \text{tr}_\mu (\langle X^{p_1} Y^{s_1} \cdots X^{p_{q_1}} Y^{s_{q_1}} \rangle \langle X^{p_{q_1+1}} Y^{s_{q_1+1}} \cdots X^{p_{q_2}} Y^{s_{q_2}} \rangle) \\ &= \sum_{\substack{\sigma, \tau \in S_{q_2} \\ aexc(\sigma) + aexc(W\tau^{-1}) - |\tau\sigma^{-1}| = q_2 + 2}} \phi(q_2, \tau\sigma^{-1}) \mathbf{Q}_\sigma^{(\mathbf{p})}(\lambda) \mathbf{Q}_{W\tau^{-1}}^{(\mathbf{s})}(\mu) \end{aligned}$$

By the same kind of argument, the following result is also established.

LEMMA 9.2.2. *For all  $\mathbf{p} = (p_1, \dots, p_q)$  and  $\mathbf{s} = (s_1, \dots, s_q)$ , one has*

$$\begin{aligned} &\lim_{N \rightarrow \infty} \text{tr}_\lambda \otimes \text{tr}_\mu (\langle X^{p_1} Y^{s_1} \cdots X^{p_q} Y^{s_q} \rangle) \\ &= \sum_{\substack{\sigma, \tau \in S_q \\ aexc(\sigma) + aexc(Z\tau^{-1}) - |\tau\sigma^{-1}| = q + 1}} \phi(q, \tau\sigma^{-1}) \mathbf{Q}_\sigma^{(\mathbf{p})}(\lambda) \mathbf{Q}_{Z\tau^{-1}}^{(\mathbf{s})}(\mu) \end{aligned}$$

### 9.3. Back to Random Matrices

The computations of the preceding section are elaborated on Lemma 2.4 of [X]. We shall recall this result below, and deduce some consequences. Let for each  $N$ , matrices  $A_1, \dots, A_q$  and  $B_1, \dots, B_q$  be such that for all families of distinct indices  $1 \leq i_1, \dots, i_k \leq q$  the limits  $\lim_{N \rightarrow \infty} \langle A_{i_1} \cdots A_{i_k} \rangle$  and  $\lim_{N \rightarrow \infty} \langle B_{i_1} \cdots B_{i_k} \rangle$  exist. For  $\sigma \in S_q$ , denote by  $\langle A \rangle_\sigma$  the limit of  $\prod_{c|\sigma} \langle A_{j_c} A_{\sigma(j_c)} \cdots \rangle$ , where the product is over the cycles of  $\sigma$  and  $j_c, \sigma(j_c), \sigma^2(j_c) \cdots$  are the elements in the cycle  $c$  of  $\sigma$ .

LEMMA 9.3.1. *One has*

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_{U(N)} \langle UA_1 U^* B_1 UA_2 \cdots UA_q U^* B_q \rangle dU \\ &= \sum_{\substack{\sigma, \tau \in S_q \\ c(\sigma) + c(Z\tau^{-1}) - |\tau\sigma^{-1}| = q+1}} \phi(q, \tau\sigma^{-1}) \langle A \rangle_\sigma \langle B \rangle_{Z\tau^{-1}} \end{aligned}$$

*Proof.* This is a consequence of Lemma 2.4 in [X], and it is proved by the same kind of arguments as we used in Section 9.2 (computations being much simpler in the present case, since the entries of the matrices commute). ■

We shall apply this result to the following situation: we have two probability measures with compact support  $m_1$  and  $m_2$ , and we choose two sequences of self-adjoint matrices  $A$  and  $B$  whose distributions in the noncommutative probability space  $(M_N(\mathbb{C}), \langle \cdot \rangle)$  converge towards  $m_1$  and  $m_2$  respectively, as  $N \rightarrow \infty$  (it is always possible to choose such sequences). In the noncommutative probability space  $L^\infty(U(N), dU) \otimes M_N(\mathbb{C})$ , let  $U = \sum_{1 \leq i, j \leq N} U_{ij} \otimes E_{ij}$  be a random, Haar distributed, unitary matrix, and consider the elements  $X = UAU^*$  and  $Y = B$ , where  $A$  and  $B$  are considered as constant matrices, then one has

LEMMA 9.3.2. *For all  $\mathbf{p} = (p_1, \dots, p_q)$  and  $\mathbf{s} = (s_1, \dots, s_q)$ , one has*

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_{U(N)} \langle X^{p_1} Y^{s_1} \cdots X^{p_q} Y^{s_q} \rangle dU \\ &= \sum_{\substack{\sigma, \tau \in S_q \\ |\sigma| + |Z\tau^{-1}| + |\tau\sigma^{-1}| = |\mathbf{p}|}} \phi(q, \tau\sigma^{-1}) \mathbf{R}_\sigma^{(\mathbf{p})} \left( \int_{\mathbb{R}} x m_1(dx), \dots, \int_{\mathbb{R}} x^{|\mathbf{p}|} m_1(dx) \right) \\ & \quad \times \mathbf{R}_{W\tau^{-1}}^{(\mathbf{s})} \left( \int_{\mathbb{R}} x m_2(dx), \dots, \int_{\mathbb{R}} x^{|\mathbf{p}|} m_2(dx) \right) \end{aligned} \quad (9.3.2)$$

*Proof.* Once we notice that  $|Z| = q - 1$ , and that the condition  $c(\sigma) + c(Z\tau^{-1}) - |\tau\sigma^{-1}| = q + 1$  is equivalent  $|\sigma| + |Z\tau^{-1}| + |\tau\sigma^{-1}| = |Z|$  this is an immediate consequence of Lemma 9.3.1 and of the definition of the polynomials  $\mathbf{R}$ . ■

Note that in this result, the  $p_j$  and  $s_j$  can take the value zero, if we agree to make the polynomials  $\mathbf{R}_\sigma$  depend on a new variable  $x_0$  and set this variable equal to 1, in the definition of  $\mathbf{R}_\sigma$ .

Observe that Lemma 9.3.1 implies that there exists a noncommutative probability space with two self-adjoint elements  $X$  and  $Y$ , whose joint distribution is such that for all  $\mathbf{p}$  and  $\mathbf{s}$ , the identity (9.3.2) is satisfied.

9.4. Connection to Freeness

We shall now relate freeness and the identity (9.3.2).

**PROPOSITION 9.4.1.** *Let  $X$  and  $Y$  be self-adjoint elements in some noncommutative probability space  $(\mathcal{A}, \varphi)$ , then  $X$  and  $Y$  are free if and only if one has, for all  $p_1, \dots, p_q$  and  $s_1, \dots, s_q$  positive integers,*

$$\begin{aligned} &\varphi[X^{p_1} Y^{s_1} \dots X^{p_q} Y^{s_q}] \\ &= \sum_{\substack{\sigma, \tau \in S_q \\ |\tau\sigma^{-1}| + |\sigma| + |Z\tau^{-1}| = |Z|}} \phi(q, \tau\sigma^{-1}) \mathbf{R}_\sigma^{(\mathbf{p})}(\varphi(X), \dots, \varphi(X^{|\mathbf{p}|})) \\ &\quad \times \mathbf{R}_{Z\tau^{-1}}^{(\mathbf{s})}(\varphi(Y), \dots, \varphi(Y^{|\mathbf{s}|})) \end{aligned} \tag{9.4.1}$$

*Proof.* We first prove sufficiency. Observe that the identity (9.4.1) determines the joint distribution of  $X$  and  $Y$ . By the remark following Lemma 9.3.2, we know that once the distributions of  $X$  and  $Y$  are specified there exists indeed a noncommutative probability space and elements  $X, Y$  satisfying these relations, moreover, the identity (9.4.1) still holds if some of the  $p_j$  or  $s_j$  are equal to 0, with the appropriate extension of the definition of  $\mathbf{R}$ . Consider now expressions of the form

$$\varphi(P_1(X) Q_1(Y) \dots P_q(X) Q_q(Y))$$

and

$$\begin{aligned} &\sum_{\substack{\sigma, \tau \in S_q \\ |\tau\sigma^{-1}| + |\sigma| + |Z\tau^{-1}| = |Z|}} \phi(q, \tau\sigma^{-1}) \\ &\quad \times \prod_{c|\sigma} \varphi(P_{j_c}(X) P_{\sigma(j_c)}(X) \dots) \prod_{c|Z\tau^{-1}} \varphi(Q_{j_c}(X) Q_{\tau(j_c)}(X) \dots) \end{aligned}$$

where the  $P_j$  and  $Q_j$  are complex polynomials in one variable, and the products are over the cycles of  $\sigma$  and  $Z\tau^{-1}$ . Both are multilinear functionals of the polynomials  $P_j$  and  $Q_j$ . Recalling the definition of the  $\mathbf{R}_\sigma$ , one sees that the two formulas above coincide when the  $P_j$  and  $Q_j$  are monomials, by the extended version of (9.4.1), it follows that they coincide for all polynomials. Consider now an expression of the form  $\varphi(P_1(X) Q_1(Y) \dots P_q(X) Q_q(Y))$  where

$$\varphi(P_j(x)) = \varphi(Q_j(Y)) = 0 \quad \text{for all } j \tag{*}$$

Let  $\sigma$  and  $\tau$  be permutations such that  $|\tau\sigma^{-1}| + |\sigma| + |Z\tau^{-1}| = |Z|$ , then by Lemma 7.1, either  $\sigma$  or  $Z\tau^{-1}$  must have a fixed point, consequently, because of the condition (\*), every term in the expansion

$$\sum_{\substack{\sigma, \tau \in S_q \\ |\tau\sigma^{-1}| + |\sigma| + |Z\tau^{-1}| = |Z|}} \phi(q, \tau\sigma^{-1}) \\ \times \prod_{c|\sigma} \varphi(P_{j_c}(X) P_{\sigma(j_c)}(X) \cdots) \prod_{c|Z\tau^{-1}} \varphi(Q_{j_c}(Y) Q_{\tau(j_c)}(Y) \cdots)$$

must vanish. In this way we have recovered the freeness condition.

The necessity of the condition is now clear since the joint distribution of a pair of free elements is determined by its one dimensional marginals. ■

**COROLLARY 9.4.2.** *The function  $\phi(q, \tau\sigma^{-1})$  is multiplicative, namely if  $q_1 < q_2$ , and one can write  $\sigma = \sigma_1\sigma_2$ ,  $\tau = \tau_1\tau_2$  where  $\sigma_1$  and  $\tau_1$  permute only elements of  $\{1, \dots, q_1\}$ , while  $\sigma_2$  and  $\tau_2$  permute only elements of  $\{q_1 + 1, \dots, q_2\}$ , then  $\phi(q, \tau\sigma^{-1}) = \phi(q_1, \tau_1\sigma_1^{-1})\phi(q_2 - q_1, \tau_2\sigma_2^{-1})$ , where we have identified  $\tau_1$  and  $\sigma_1$  (respectively  $\tau_2$  and  $\sigma_2$ ), with elements of  $S_{q_1}$  (respectively  $S_{q_2 - q_1}$ ).*

*Proof.* Formula (9.4.1) gives us an explicit way of writing the joint moments of two free random variables knowing the moments of each of them. We can compare this formula with the one based on non-crossing partitions in [Sp2], namely formula (2.5) above, where the cumulants are replaced with their defining formula. The permutations  $\sigma$ ,  $\tau$  and  $\tau\sigma^{-1}$  occurring in the formula (9.4.1) satisfy the condition  $|\sigma| + |Z\tau^{-1}| + |\tau\sigma^{-1}| = |Z|$ . As we recalled in Section 2.7, this implies that the partitions of  $\{1, \dots, q\}$  induced by their cycle decompositions are non-crossing partitions. Comparison with the formula for the moments (2.5) shows that in fact  $\phi(q, \tau\sigma^{-1})$  is equal to the Möbius function on  $NC$ , evaluated on  $\iota^{-1}(\tau\sigma^{-1})$ . The multiplicativity of  $\phi$  then follows from the multiplicativity of the Möbius function. ■

## 10. END OF THE PROOF OF THEOREM 3.1.3

### 10.1. Preliminary Estimates

From Proposition 7.6 we deduce the following

**LEMMA 10.1.** *For each integer  $n > 0$  there exists a constant  $K_n$  such that for all Young diagrams  $\lambda \in \mathcal{Y}$ , all permutations  $\sigma \in S_q$ , and all  $\mathbf{p}$  with  $|\mathbf{p}| \leq n$ , one has*

- (1)  $|\mathbf{R}_\sigma^{(\mathbf{p})}(\lambda)| \leq \text{width}(\lambda)^{|\mathbf{p}|}$
- (2) If  $aexc(\sigma) + |\sigma| > q$ , then  $|\mathbf{Q}_\sigma^{(\mathbf{p})}(\lambda)| \leq K_n \text{width}(\lambda)^{|\mathbf{p}|-1}$
- (3) If  $aexc(\sigma) + |\sigma| = q$ , then  $|\mathbf{Q}_\sigma^{(\mathbf{p})}(\lambda) - \mathbf{R}_\sigma^{(\mathbf{p})}(\lambda)| \leq K_n \text{width}(\lambda)^{|\mathbf{p}|-1}$

10.2. *Computation of the Mean*

We now prove that the mean value of the measure  $m_\nu$ , for the  $[v]$  occurring in the decomposition of  $[\lambda] \circ [\mu]$  for large diagrams, is close to  $m_\lambda \boxplus m_\mu$ .

LEMMA 10.2. *For all  $A > 1$ , and  $m$  a positive integer, one has*

$$\begin{aligned} \sum_{v \in \mathcal{O}_{|\lambda|+|\mu|}} c_{\lambda\mu}^v \frac{\dim([v])}{\dim([\lambda] \circ [\mu])} \int_{\mathbb{R}} x^k m_\nu(dx) \\ = \int_{\mathbb{R}} x^k m_\lambda \boxplus m_\mu(dx) + O(q^{k/2-1/2}) \end{aligned}$$

uniformly over all Young diagrams  $\lambda, \mu$  satisfying  $\text{width}(\lambda), \text{width}(\mu) \leq A\sqrt{q}$  and all  $k \leq m$ .

*Proof.* By Proposition 7.3 one has

$$\sum_{v \in \mathcal{O}_{|\lambda|+|\mu|}} c_{\lambda\mu}^v \frac{\dim([v])}{\dim([\lambda] \circ [\mu])} \int_{\mathbb{R}} x^k m_\nu(dx) = \lim_{N \rightarrow \infty} \text{tr}_\lambda \otimes \text{tr}_\mu (\langle (X + Y)^k \rangle)$$

We can expand  $(X + Y)^k$  in the right hand side of the expression above, and then use 9.2.2 to evaluate the limit of each term when  $N \rightarrow \infty$ . We get a sum of terms of the form  $\phi(q, \tau\sigma^{-1}) \mathbf{Q}_\sigma^{(\mathbf{p})}(\lambda) \mathbf{Q}_{Z\tau^{-1}}^{(\mathbf{s})}(\mu)$  with  $|\mathbf{p}| + |\mathbf{s}| = k$ . By Lemma 10.1, we can replace each term  $\mathbf{Q}$  by either 0 or the corresponding  $\mathbf{R}$ , depending on whether condition (3) or (2) of Lemma 10.1 is satisfied, making an error which is bounded by some constant times  $q^{k/2-1/2}$ . By Lemma 9.4.1, the result of this replacement gives exactly  $\varphi((\tilde{X} + \tilde{Y})^k)$  where  $\tilde{X}$  and  $\tilde{Y}$  are free random variables in some noncommutative probability space  $(\mathcal{A}, \varphi)$ , with distribution  $m_\lambda$  and  $m_\mu$ . This is  $\int_{\mathbb{R}} x^k m_\lambda \boxplus m_\mu(dx)$  by definition of free convolution. ■

10.3. *Estimation of the Variance*

We shall estimate the variance of the expression  $\int_{\mathbb{R}} x^k m_\nu(dx)$ .

LEMMA 10.3.1. *Fix  $A > 1$ , and  $m$  a positive integer, then one has*

$$\begin{aligned} & \lim_{N \rightarrow \infty} \operatorname{tr}_\lambda \otimes \operatorname{tr}_\mu (\langle X^{p_1} Y^{s_1} \dots X^{p_{q_1}} Y^{s_{q_1}} \rangle \langle X^{p_{q_1+1}} Y^{s_{q_1+1}} \dots X^{p_{q_2}} Y^{s_{q_2}} \rangle) \\ &= \lim_{N \rightarrow \infty} \operatorname{tr}_\lambda \otimes \operatorname{tr}_\mu (\langle X^{p_1} Y^{s_1} \dots X^{p_{q_1}} Y^{s_{q_1}} \rangle) \\ & \quad \times \lim_{N \rightarrow \infty} \operatorname{tr}_\lambda \otimes \operatorname{tr}_\mu (\langle X^{p_{q_1+1}} Y^{s_{q_1+1}} \dots X^{p_{q_2}} Y^{s_{q_2}} \rangle) \\ & \quad + O(q^{|\mathbf{p}|/2 + |\mathbf{s}|/2 - 1/2}) \end{aligned}$$

uniformly over all Young diagrams  $\lambda, \mu$ , satisfying  $\operatorname{width}(\lambda), \operatorname{width}(\mu) \leq A\sqrt{q}$  and  $\mathbf{p} = (p_1, \dots, p_{q_2}), \mathbf{s} = (s_1, \dots, s_{q_2})$  with  $|\mathbf{p}|, |\mathbf{s}| \leq m$ .

*Proof.* By Lemma 9.2.1, one has

$$\begin{aligned} & \lim_{N \rightarrow \infty} \operatorname{tr}_\lambda \otimes \operatorname{tr}_\mu (\langle X^{p_1} Y^{s_1} \dots X^{p_{q_1}} Y^{s_{q_1}} \rangle \langle X^{p_{q_1+1}} Y^{s_{q_1+1}} \dots X^{p_{q_2}} Y^{s_{q_2}} \rangle) \\ &= \sum_{\substack{\sigma, \tau \in S_{q_2} \\ \operatorname{aexc}(\sigma) + \operatorname{aexc}(W\tau^{-1}) - |\tau\sigma^{-1}| = q_2 + 1}} \phi(q_2, \tau\sigma^{-1}) \mathbf{Q}_\sigma^{(\mathbf{p})}(\lambda) \mathbf{Q}_{W\tau^{-1}}^{(\mathbf{s})}(\mu) \end{aligned}$$

Applying Lemma 10.1 we can replace each occurrence of a term  $\mathbf{Q}$  by either 0 or the corresponding  $\mathbf{R}$ , depending on the permutation, making an error bounded by  $O(q^{|\mathbf{p}|/2 + |\mathbf{s}|/2 - 1/2})$ . Taking care of the conditions on the contributing permutations, we thus obtain the quantity

$$\Gamma = \sum_{\substack{\sigma, \tau \in S_{q_2} \\ |\sigma| + |W\tau^{-1}| + |\tau\sigma^{-1}| = |W|}} \phi(q_2, \tau\sigma^{-1}) \mathbf{R}_\sigma^{(\mathbf{p})}(\lambda) \mathbf{R}_{W\tau^{-1}}^{(\mathbf{s})}(\mu)$$

Let  $\mathbf{p}_1 = (p_1, \dots, p_{q_1}), \mathbf{s}_1 = (s_1, \dots, s_{q_1}), \mathbf{p}_2 = (p_{q_1+1}, \dots, p_{q_2}), \mathbf{s}_2 = (s_{q_1+1}, \dots, s_{q_2})$ . We do the same in the expressions for

$$\lim_{N \rightarrow \infty} \operatorname{tr}_\lambda \otimes \operatorname{tr}_\mu (\langle X^{p_1} Y^{s_1} \dots X^{p_{q_1}} Y^{s_{q_1}} \rangle)$$

and

$$\lim_{N \rightarrow \infty} \operatorname{tr}_\lambda \otimes \operatorname{tr}_\mu (\langle X^{p_{q_1+1}} Y^{s_{q_1+1}} \dots X^{p_{q_2}} Y^{s_{q_2}} \rangle)$$

applying Lemma 9.2.2, and we get two quantities

$$\Gamma_1 = \sum_{\substack{\sigma, \tau \in S_{q_1} \\ |\sigma| + |Z_1\tau^{-1}| + |\tau\sigma^{-1}| = |Z_1|}} \phi(q_1, \tau\sigma^{-1}) \mathbf{R}_\sigma^{(\mathbf{p}_1)}(\lambda) \mathbf{R}_{Z_1\tau^{-1}}^{(\mathbf{s}_1)}(\mu)$$

where  $Z_1 = (12 \cdots q_1)$ , and an analogous formula for  $\Gamma_2$ . We find that the sum defining  $\Gamma$  is over permutations  $\sigma, \tau$  of  $\{1, \dots, q_2\}$  satisfying  $|\sigma| + |W\tau^{-1}| + |\tau\sigma^{-1}| = |W|$ . This condition is an equality in the triangular inequality (satisfied by all permutations)  $|\sigma| + |W\tau^{-1}| + |\tau\sigma^{-1}| \geq |W|$ . It implies that the partitions of  $\{1, \dots, q_2\}$  induced by the cycles of the permutations  $\sigma$  and  $\tau$  are finer than the one induced by  $W$  (see Section 2.7). Thus all permutations  $\sigma$  and  $\tau$  appearing in the sum can be decomposed as  $\sigma = \sigma_1\sigma_2$  and  $\tau = \tau_1\tau_2$  where  $\sigma_1$  and  $\tau_1$  permute only elements of  $\{1, \dots, q_1\}$ , while  $\sigma_2$  and  $\tau_2$  permute only elements of  $\{q_1 + 1, \dots, q_2\}$ . By Corollary 9.4.2, one has  $\phi(q_2, \tau\sigma^{-1}) = \phi(q_1, \tau_1\sigma_1^{-1})\phi(q_2 - q_1, \tau_2\sigma_2^{-1})$ , moreover one has  $\mathbf{R}_\sigma^{(\mathbf{p})}(\lambda) = \mathbf{R}_{\sigma_1}^{(\mathbf{p}_1)}(\lambda)\mathbf{R}_{\sigma_2}^{(\mathbf{p}_2)}(\lambda)$ , and it follows that  $\Gamma = \Gamma_1\Gamma_2$ . ■

Define  $\sum_{\nu \in \mathcal{Y}_{|\lambda|+|\mu|}} c_{\lambda\mu}^\nu \dim([\nu])/\dim([\lambda] \circ [\mu]) \int_{\mathbb{R}} x^k m_\nu(dx) = M(\lambda, \mu, k)$  for Young diagrams  $\lambda, \mu$ , and  $k \geq 1$  integer.

As in Sections 4 and 5, we deduce from 10.2 and 10.3.1

**COROLLARY 10.3.2.** *For all  $A > 1$  and integer  $m \geq 1$ , one has*

$$\sum_{\nu \in \mathcal{Y}_{|\lambda|+|\mu|}} c_{\lambda\mu}^\nu \frac{\dim([\nu])}{\dim([\lambda] \circ [\mu])} \left( \int_{\mathbb{R}} x^k m_\nu(dx) - M(\lambda, \mu, k) \right)^2 = O(q^{k-1/2})$$

uniformly over all Young diagrams  $\lambda, \mu$  satisfying  $\text{width}(\lambda), \text{width}(\mu) \leq A\sqrt{q}$  and all  $k \leq m$ .

Theorem 3.1.3 now follows from Lemma 10.2 and Corollary 10.3.2 by an application of Markov inequality.

Theorem 1.6.2 again can be proved along the lines of Theorems 1.4.2 and 1.5.2.

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